# THE ASSOCIATION AND ITS SECTIONS

By H. E. SLAUGHT, University of Chicago

It must be a source of great satisfaction to all those who have closely observed the development of the Mathematical Association of America during the first decade of its history to see the steady and substantial development of its Sections. It will be recalled that Ohio and Missouri were contestants for the honor of securing the first charter for a section and that Ohio won by the margin of a few minutes, both petitions being presented within an hour after the final adoption of the constitution at the organization meeting of the Association in Columbus, Ohio, in December, 1915.

Since that time the number of sections has grown to seventeen, the latest being the Philadelphia Section intended to serve about one hundred members residing in contiguous portions of Pennsylvania, New Jersey and Delaware. All of the sections hold yearly meetings and some of them twice yearly with an average attendance of about twenty-five members, the highest numbers recently recorded being 51, 32, 31, 29, . . . . and the lowest 11, 16, 17, 21 . . . . .

Aside from members the attendance is usually swelled by about one-half, consisting of visitors from allied departments and students of mathematics in the institution acting as host. The presence of this latter group is especially important in that it forms a bond of union between the Association and the clientéle from which future recruits for mathematics are to be drawn. Moreover the president or dean of the host institution usually welcomes the visitors and attends the dinner thus adding significance to the occasion and giving prominence to mathematics among all the institutions represented. Indeed, the host institution, or some organization in the town not infrequently gives a complimentary dinner or luncheon to the Section—and the local papers give corresponding publicity. For example, at the recent meeting of the Louisiana-Mississippi Section, the Shreveport Chamber of Commerce gave a complimentary dinner to over ninety persons consisting of members of the Section and of secondary teachers who had met to organize a Branch of the National Council. The Shreveport *Journal* gave nearly three columns of publicity to the various phases of the meeting.

During the ten years, the sections, as fast as organized, have "carried on" *continuously*, with only *two* exceptions. In each of these cases an interruption of one year occurred owing to failure of officers to function efficiently. It is believed now that in both of these cases the proper stimulus has been applied and the result will be continuity of action from now on. A study of the programs of the section meetings as reported in the MONTHLY is interesting and significant.

An average of about seven papers have been presented at the meetings during 1925 and 1926, the lowest number being five and the highest nineteen. In some cases discussion has centered around local conditions such as state requirements in the high schools and admission to the colleges and state universities. In other cases valuable expository papers have been presented on various topics. In still other cases new processes or new results have been exhibited. An inspection of these programs affords an illuminating cross-section of the activities of the sections. And not the least important feature of these meetings is a by-product in the form of mutual acquaintanceship among colleagues and mutual exchange of views among those engaged in similar work.

If we measure the influence of the Association by the attendance of its members at meetings and the presentation of papers, we see at once the importance of the sections. At the two national meetings the total attendance has been around 250 (it was over 300 in 1926 because of the record attendance at Philadelphia), while the total attendance at sectional meetings has been over 400. At the two national meetings about 16 papers (total) are usually presented while at the sectional meetings over 100 papers are presented yearly. If, in addition, we consider the cumulative and combined influences resulting from seventeen sectional meeting places and host institutions as against two for the national meetings, we see again the importance of the sections. Moreover, there is little prospect that the attendance at national meetings will become much larger than at present owing to the great distances to such centers from the more remote parts of the country, whereas it is not impossible that the total attendance at section meetings may be greatly increased—even doubled or trebled—as the interest continues to grow,—especially in view of the fact that stone roads are rapidly spreading to all parts of the country so that more and more members may reach their sectional meeting places in their own conveyances or in public busses.

Naturally, the question arises: What can we do to increase the scope and usefulness of the sections of the Association? One answer to this question is to assist them in improving their present activities wherever such assistance is needed. Some of the sections are much more favorably located than others. Some have a far larger quota of members than others. Some have a much larger geographical distribution than others. Evidently the Association should do all in its power to encourage the sections which are laboring under disadvantages. For instance, it is a great stimulus to those sections to have some outstanding representative of the Association attend their meetings. This has been done occasionally—sometimes at the expense of the section, sometimes at the expense of the Association and sometimes dividing the expense between them. Professor Coolidge, when president, proposed that the Association should send a delegate to every section meeting, but the matter was held in

abeyance because the funds of the Association were not sufficient to finance such a project. The question was again discussed at the Philadelphia meeting and a special committee was appointed to consider it in connection with the related question of giving expository lectures at section meetings. As a start in this direction it was agreed that it would be very stimulating to the sections if each author of a Carus Monograph could deliver one or more lectures on his subject at each of the section meetings, or at least at a number of such meetings especially of those sections which are most disadvantageously located. It is hoped that the Carus Fund may eventually be adequate to finance this procedure and that such a beginning may grow into a more extended series of expository lectures each year before the sections.

Another answer to the question of increasing the usefulness of the sections is to give them some important objective toward which to direct their energies. Such an objective was also suggested by Professor Coolidge when he negotiated the affiliation agreement between the Association and the New England Association of Teachers of Mathematics, an organization concerned chiefly with the secondary field. By this action our Association reaffirmed its vital interest in secondary education, having already emphatically proclaimed that fact when, at the very beginning of its career, it sponsored the National Committee on Mathematical Requirements and actually appointed the chairman, Professor J. W. Young, and the original nucleus of six members of the committee with power to enlarge their number. The report of that committee, everywhere recognized as an Association product, after being distributed to more than 25,000 teachers is still in active demand and has probably produced a more far-reaching effect on secondary teaching than any document of its kind in this generation. With the completion of the work of this committee, the responsibility for carrying on its spirit and influence seemed to fall upon the National Council of Teachers of Mathematics, a federation of secondary organizations, one of which is the above mentioned New England Association.

The foregoing statement of facts throws into clear light the two-fold opportunity and responsibility of the Association. On the one hand it stands squarely in the collegiate field for better teaching, for better mathematical understanding, for wider dissemination of mathematical knowledge and interest through the medium of exposition, especially as embodied in the idea of the Carus Monographs, and for the beginnings of research so essential to the highest success in teaching and so important a stimulus to later scientific activity. In all of these respects the Association has held to its ideals throughout the past decade and by its close cooperation with the Annals of Mathematics and with the American Mathematical Society, as well as in many other ways, it has, by common consent, contributed strongly to the general forward movement of mathematical interests in this country. On the other hand the Associ-

ation, as already explained, has used its strong influence for the betterment of mathematical conditions in the secondary field, not by usurping authority nor by supplanting any of the organizations in that field, but by acting as adviser and friend to their representatives and sitting with them to work out in tolerant cooperation the best solution of the difficult problems which confronted that National Committee. It was, indeed, an inspiring sight to observe the sessions of that body composed, as it was, of university, college and secondary teachers, as they together weighed and tested every idea or plan that was presented.

And now, in the same fraternal spirit, it has occurred to some of us that an opportunity is open for cooperation between the sections of the Association and the Branches of the National Council. It would seem that meetings of these two organizations held at the same time and place, the sessions extending over a Friday and Saturday, with a joint dinner on Friday evening, would result in definite advantages to both groups. In particular it would give each group an opportunity to become acquainted with difficulties confronting the other and to exchange suggestions and remedies. But most important, perhaps, would be the opportunity for mutual acquaintanceship between the two groups and hence for mutual understanding and sympathy. So obvious seem the possibilities of such a scheme that the Trustees at Philadelphia voted to approve a tentative plan proposed for the Louisiana-Mississippi Section as follows: (1) To urge all secondary teachers of mathematics in these two states to meet with the Section at Shreveport on March 4–5 for the purpose of forming a Branch of the National Council. (2) To invite any secondary teachers who might wish to join the Association to do so without the customary two dollar initiation fee, provided they are already members of the National Council and are thus using the two dollars for subscriptions to the *Mathematics Teacher*, the official journal of the Council. This agreement is the same as that already existing with the New England Association, except that we have no section of our Association in New England and hence can meet with them only when the national meetings are held in New England territory.

Great credit is due to Professor S. T. Sanders of Louisiana State University, Chairman of the Louisiana-Mississippi Section of the Association, for the effective way in which he directed the publicity campaign for the meeting. The attendance of secondary teachers was good but not at all commensurate with the effort put forth in their behalf. Nevertheless, considering the fact that the National Council had heretofore been little known to most of them, it was pleasant to observe the serious and earnest spirit of those present and the effective way in which they proceeded to perfect their organization. They were doubtless inspired by the presence of Miss Marie Gugle, assistant superintendent of schools, Columbus, Ohio, president of the National Council, and Professor W. D. Reeve of Teacher's College, Columbia University, joint editor of the

Mathematics Teacher, both of whom took prominent parts in the program. The Association was also represented by Professor H. E. Slaught as official delegate appointed by the Trustees. It seems certain that the influence of such a meeting must result in the widening and deepening of mathematical interest in those two states. Professor Sanders was reelected chairman of the Section and he proposes to continue the good work during the coming year.

At least one other section has been thinking of a similar cooperation with the National Council and still others may wish to consider its desirability. Also there are other secondary organizations in locations where there are no sections of the Association, some of which might wish to affiliate with us as in the case of the New England Association. Again there are portions of the country where new sections of the Association might well be established. All of these possibilities seem destined to be tried out as the Association grows in power and influence. Finally, the sections have a great responsibility for maintaining and steadily increasing the membership of the Association. A good slogan for any section would be: "One hundred percent membership," meaning that every person engaged in teaching mathematics in any collegiate institution within the territory of the section should be a member of the Association. This slogan might well apply also to institutional memberships. The total membership (individual and institutional) is now well over 2,000 but this slogan fully realized in all the sections would swell the number to 3,000 or more, and correspondingly increase the potential activities.

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## THE FIFTH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fifth regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of California at Los Angeles, on Saturday, March 12, 1927. Professor H. C. Willett presided.

There were fifty present, inculding the following thirty-four members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, H. Bateman, Clifford Bell, E. T. Bell, R. W. Bolton, W. D. Cairns, J. R. Campbell, M. Collier, P. H. Daus, J. B. Ernsberger, H. H. Gaver, H. E. Glazier, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, G. R. Livingston, W. E. Mason, A. McClellan, B. Podolsky, L. E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, M. Skarstedt, D. V. Steed, F. C. Touton, H. C. Van Buskirk, H. C. Willett, Clyde Wolfe, E. R. Worthington.

The meeting began with a luncheon at the Newman Club. Professor W. D. Cairns, Secretary-Treasurer of the Association, was the guest of the Section, and after the luncheon, told us something of the aims and plans of the Association.

The following officers were elected for the coming year: Professor W. P. RUSSELL, chairman, Professor E. T. BELL, vice chairman, Professors O. W. ALBERT and L. D. AMES, program committee.

The program consisted of the following papers. Short abstracts appear below.

1. "Symmetric functions in the algebra of logic," by Mr. MORGAN WARD, California Institute of Technology.

2. "On the concepts of area and volume," by Professor E. R. HEDRICK, University of California at Los Angeles.

3. "A matrix notation for $m$-dimensional geometry," by Professor L. D. AMES, University of Southern California.

4. "Unsolved problems in the theory of periodic functions," by Professor E. T. BELL, California Institute of Technology.

5. "A set of construction problems related to the problem of Apollonius," by Professor P. H. DAUS, University of California at Los Angeles.

6. "Functions of closest approximation on an infinite range," (by invitation), by Professor W. D. CAIRNS, Oberlin College.

1. Mr. Ward defined the logical symbols $A+B$ and $A\times B$ with reference to classes and explained the fundamental operations in such an algebra. He proceeded to discuss the relations between the elementary symmetric functions of the roots of an equation in this algebra, indicating how serial order can be defined. He showed the relations between the symmetric functions of the roots of two equations and indicated how they led to certain problems in Diophantine analysis.

2. Professor Hedrick discussed a set of axioms concerning area, which force one to the Riemannian definition of area. They are: (1) The area of a rectangle is the product of the base and its altitude; (2) The area of a figure formed by a finite number of rectangles is the sum of their areas; (3) Of two regions, of which one is interior to the other, the inner one has not the greater area. He indicated how these axioms assume too much, and how they may be replaced by assumptions which enable one to apply these concepts to some of our modern definitions of space, as for example, a Fréchet space.

3. Professor Ames considered the matrix of $m$ points $(x_i, y_i, z_i, \cdots)$. If the rank of the matrix is 1, 2, 3, $\cdots$, then the matrix represents a point, line, plane, etc. He indicated how certain problems could be conveniently solved using this notation.

4. Professor Bell discussed several problems in the theory of multiply-periodic functions, which are unsolved in the sense that the so-called solutions refer to problems in another branch of mathematics for which we do not have solutions. Among these problems are the following: (1) How many solutions does the Diophantine equation $m=x_1^2+\cdots+x_r^2$ have? If $r<8$ and odd, the answer

is a complicated transcendental function, which is a mathematical but not a practicable solution. (2) Weierstrass's first problem, that is, to expand $sn(u, K^2)$ in a power series in $u$. The coefficients are polynomials in $K$, which are considered as known, when their determination has been referred to a problem in linear difference equations. He also discussed the evolution of certain theta functions, and the problem of expanding doubly-periodic functions in two variables, indicating how the so-called solutions were not really solutions.

5. This paper will appear in the MONTHLY.

6. Professor Cairns extended to the interval $(-\infty, \infty)$, a theorem by Professor Dunham Jackson given in Vol. 22 of the Transactions of the American Mathematical Society. If $f(x)$ is continuous for finite $x$ and vanishes at $\pm\infty$ like $|x|^n e^{-x^2/2}$, $n$ a definite positive number, there exists a set of constants $c_i$, for which we may approximate to $f(x)$ by $\phi(x)$, a linear combination of continuous and linearly independent functions $p_i(x)$ and limited at $\pm\infty$ like $f(x)$, the approximation being such that the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\left[\frac{f(x)-\varphi(x)}{e^{-x^2/4}}\right]^m dx$$

attains a minimum, whenever $m>1$.

P. H. DAUS, *Secretary*

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## THE TWENTIETH REGULAR MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The twentieth regular meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held on Saturday, December 4, 1926 at the United States Naval Academy, Annapolis, Md., the morning session opening at 11 A.M. and the afternoon session at 2 P.M. Those attending the meeting were guests of the Annapolis members at luncheon. Chairman J. A. Bullard presided at both sessions.

There were 60 present, including the following 41 members of the Association. O. S. Adams, R. N. Ashmun, H. G. Avers, W. W. Bigelow, G. A. Bingley, L. M. Blumenthal, C. C. Bramble, J. A. Bullard, P. Capron, J. R. Clements, J. A. Duerksen, J. B. Eppes, P. J. Federico, G. L. Fentress, M. Goldberg, W. M. Hamilton, G. W. Hansen, H. P. Kaufman, A. E. Landry, C. L. Leiper, Florence P. Lewis, E. S. Mayer, F. Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, E. C. Philips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, J. A. Schad, J. T. Spann, T. H. Taliaferro, A. A. Tomeldon, Marion M. Torrey, J. Tyler, J. E. Willis.

The program follows, accompanied by abstracts of the papers:

1. "The theorems of Menelaus and Ceva and their extension," by Dr. P. WERNICKE, U. S. Patent Office, by invitation.

2. "Invariants under rotation" by Professor F. MORLEY, Johns Hopkins University.

3. "Numerical integration" by Assistant Professor L. M. KELLS, United States Naval Academy.

4. "Modern navigation" by Commander A. M. R. ALLEN, United States Naval Academy, by invitation of the section.

5. "The Byrd polar flight" by H. G. AVERS, United States Coast and Geodetic Survey.

6. "A problem in maxima and minima" by Professor PAUL CAPRON, United States Naval Academy.

7. "A monogram designed to simplify certain geodetic calculations" by H. S. RAPPLEYE, United States Coast and Geodetic Survey, by invitation.

8. "Functions formed by continuous substitution" by Professor JOHN TYLER, United States Naval Academy.

1. Two triangles in a plane determine by the joins of their corresponding vertices an "imperspective central triangle" and by the intersections of corresponding sides an "imperspective axial triangle." Calling the product of the ratios in which one triangle is divided by the joins of vertices, its "menelaus," and the product of the ratios in which its sides are divided by the intersections with those of the other triangle, its "ceva," the menelaus of either triangle is proved equal to the negative ceva of the other. In the case of perspective, the menelai become unity (theorem of Menelaus) and the cevas, negative unity (theorem of Ceva). A similar theorem is proved for two quadrilaterals, or "tetragrams" in three-dimensional space, which define tetragons by their vertices, and tetrahedra by the planes drawn through consecutive sides.

2. The expression $x_1+\epsilon x_2+\epsilon^2 x_3+\cdots+\epsilon^{n-1}x_n$, where $\epsilon^n=1$, is called a Lagrange resolvent of the equation whose roots are $x_i$. See Pascal, *Repertorium*, 2nd edition, vol. 1, p. 307. Regarding the roots as points of a plane, so ordered as to form a polygon and with a definite starting point $x_1$, the expressions become attached to the polygon. Under the translations $x_i=y_i+b$ where $\epsilon\neq 1$, the expression is an invariant. We then have $n-1$ Lagrange invariants (under translation):

$$v_1 = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \cdots; \; v_2 = x_1 + \epsilon^2 x_2 + \epsilon^4 x_3 + \cdots;$$

where $\epsilon=\exp 2i\pi/n$. If we add to these $v_0=x_1+x_2+x_3+\cdots$ which is not an invariant, we can readily solve for the $x_i$ in terms of the $v_i$ and hence show that the above invariants are a complete system. To illustrate the theory the norm and area of a hexagon were expressed explicitly in terms of Lagrange invariants.

3. The paper on numerical integration illustrated one method of procedure by finding several points on that solution of $dy/dx=x+y$ which passes through point (0, 1), derived a trial formula and an improvement formula, showed how

the method could be applied to the trajectory of an object shot through the air, and discussed briefly several different methods of procedure.

4. Commander Allen discussed the contributions of various nations to the problems of modern navigation. The methods of teaching navigation at the United States Naval Academy were presented. A display of literature on the subject and models to illustrate problems were on exhibit. Upon completion of the paper, the members were at leisure to examine the exhibit and obtain information on desired points.

5. Mr. Avers spoke on the scientific preparation for the Byrd polar flight, the methods of taking observations during the flight and the computation of the observations to determine when Byrd was exactly over the pole. He also spoke concerning the committee who were appointed to re-check Byrd's data.

6. Gores are cut from the corners of a square sheet of tin and a strip folded up along each edge so that a container with sloping sides is formed. The problem required the maximum capacity to be obtained. The interest in the problem lay in the necessity for unusual care in its treatment and in the fact that the capacity found was exactly 7/6 as great as that of a container formed when the gores become squares and the container is rectangular (1.16667 to five decimals). Professor Capron also developed that the breadth of the strip is almost exactly a mean proportional between the half edge of the sheet and the length cut off from the edge at each corner, so that a correspondingly simplified problem in one variable gives a maximum capacity within 0.001 of 1% of the capacity desired, and an approximate solution of the original problem depending only upon a quadratic surd ($\sqrt{3}$) is within 0.003 of 1%.

Characteristics were also mentioned of the corresponding problem for a rectangular sheet.

7. Mr. Rappleye discussed a nomogram designed to give the reduction of horizontal directions to sea level. Copies of the nomogram were given to the members.

8. Professor Tyler discussed a class of functions formed by continuous substitution. Thus if $f(t)$ be a function of $t$ and if we replace $t$ by $f(t)$, the result is denoted by $f^2(t)$. If the argument $t$ be replaced by $t$, $x-1$ times, we would have the expression $f^x(t)$. He then defined $f^x(t)$ for all values of $x$, real or complex by the relation $f^x(t)=r^{-1}[x+r(t)]$ where $r(t)$ is a function of $t$ and $r^{-1}(t)$ its inverse function. The relation between $f$ and $r$ is given by the relation $f(t)=r^{-1}[1+r(t)]$. From this definition the index law of $f^x[f^y(t)]=f^{x+y}(t)$ follows and $f^0(t)=t$, also $f^{-x}(t)$ is the inverse function of $f^x(t)$. He also showed that if $x$ is the period of $r^{-1}(t)$, then $f^x(t)$ will repeat under substitution for every integral multiple of $x$. Various illustrations of this class of functions were given.

J. R. MUSSELMAN, *Secretary*

# THE DUTY OF EXPOSITION WITH SPECIAL REFERENCE TO THE CAUCHY-HEAVISIDE EXPANSION THEOREM[1]

By FRANCIS D. MURNAGHAN, Johns Hopkins University

As the speaker representing the Mathematical Association of America at this session I propose this morning to call your attention to one of the more important duties of a mathematician, namely, the duty of explaining as clearly as possible mathematical truths and discoveries both to his fellow mathematicians and to students of mathematics in general. The American Mathematical Society is especially devoted to the encouragement of research and the secretary of that society has called to your attention during this meeting the remarkable increase in the number of papers presented annually to the Society during the past five years. This increase has been such as to make the problem of publishing the results of mathematical research in America a very acute and difficult one. At the same time it has become more desirable than ever before that the papers published in our American journals should be as clear and as easily readible as possible. The Mathematical Association of America is especially concerned with the teaching and exposition of mathematical truth and it is pretty generally agreed that it is highly desirable that care should be taken to make this teaching and exposition to students as good as possible. I fear, however, that some of my friends who are particularly interested in research agree to this in a rather condescending manner; their tone implying that, whilst nothing should be done to discourage anyone beginning the study of mathematics, such care is not so necessary nor even so desirable when writing for fellow mathematicians. The underlying idea is that a competent mathematician usually prefers to glance at a paper, see the results arrived at, and then derive these results in his own individual manner. I believe that this opinion is not justified by the facts and in support of this belief I shall mention two instances which have recently come to my attention where mathematicians of great competence failed, through a lack of detail or of clarity in available expositions of known results, to arrive at immediate and important corollaries of these results. I shall be happy if my talk this morning tends to make the editors of our mathematical journals insist more definitely on clearness of exposition when considering papers submitted for publication. One can surmise that the writers of at least some of our papers set down their results with the referee of the paper more in mind than the prospective readers. They neglect, therefore, to state or emphasize points which they think will be familiar to the referee. Of course this is unfortunate since there is usually a one-to-one

[1] Read at the joint session of Section A of the American Association for the Advancement of Science, the American Mathematical Society and the Mathematical Association of America, Philadelphia, Dec. 30, 1926.

correspondence between the writer of a paper and his referee and it is the proper function of an editor of a mathematical journal to make the correspondence between the writers of the papers appearing in that journal and their readers as multiform, in a one-sided way, as possible.

It is a good rule of life to have a vocation and an avocation. My own particular vocation is the study and teaching of the applications of mathematics. The study of pure geometry serves as an avocation and I shall, therefore, rather naturally choose my first illustration from the realm of pure geometry reserving for my closing illustration a rather important result in applied mathematics.

One of the problems of pure geometry possessing an interesting history is the Torricelli or Fermat problem of finding a point in a plane having the property that the sum of the three distances from it to three given points in the plane is a minimum.[1] Every American student of mathematics is familiar with the discussion of this problem given in the Goursat-Hedrick *Mathematical Analysis*. However, as Darboux once said, an elegant demonstration touching the heart of the matter and nothing else is often of more interest to a geometer than the result itself and such a solution of the problem in question was given by Steiner.[2] He pointed out that, since the sum of the perpendiculars from any point inside an equilateral triangle upon the three sides is constant, all one has to do is to find a point $P$ such that the perpendiculars from the three given points $A, B, C$ to $PA, PB, PC$, respectively, form an equilateral triangle; in the technical language of the geometer the anti-pedal triangle of $P$ with respect to the triangle $ABC$ is to be equilateral. In fact, if $P'$ is any other point, each of the distances $P'A, P'B, P'C$ is greater than, or at least equal to, the corresponding perpendicular distance $P'A', P'B', P'C'$ from $P'$ to a side of the equilateral triangle whose sides pass through $A$, $B$ and $C$. Hence $P'A+P'B+P'C \geqq P'A'+P'B'+P'C'$, the equality sign holding only when $P'$ coincides with $P$. But $P'A'+P'B'+P'C'=PA+PB+PC$ so that $P'A+P'B+P'C>PA+PB+PC$ if $P'$ is distinct from $P$. The construction of $P$ follows at once on observing that each of the angles $BPC, CPA, APB$ is 120°. This is all delightful but now comes the point where Steiner failed as an expositor. He remarks in a note that the result can easily be extended to space. What was the consequence of this economy of paper? The famous geometer Sturm, whose four volume treatise on synthetic geometry[3] is familiar to every student of that subject,

[1] For historical references see the article by M. Zacharias in the Encyclopädie der mathematischen Wissenschaften, vol. III–1, p. 1129. See also this Monthly, vol. 24 (1917), pp. 42–44, 243–244 and vol. 30 (1923), pp. 127–131.

[2] J. Steiner, Werke, vol. 2, p. 93 and p. 729.

[3] R. Sturm, *Die Lehre von den Geometrischen Verwandtschaften* (Teubner, 1908–1909).

considered in 1884[1] the problem of finding a point in space having the property that the sum of the four distances $PA$, $PB$, $PC$, $PD$ to four points $A$, $B$, $C$, $D$ in space is a minimum. He treated the problem again on page 67 of his extremely interesting book on *Maxima und Minima in der Elementaren Geometrie* (Teubner, 1910), and finally returned to it in 1913 in a paper in Crelle's journal[2] in which he says as his final remark: "Aber konstruierbar ist er nicht, weil er Schnittpunkt von drei Flächen höherer Ordnung ist." Nevertheless Steiner's remark that the method given for the plane problem is aplicable to the space problem is justifiable if the proper viewpoint is taken. Instead of focussing attention on the fact that the three angles $BPC$, $CPA$, $APB$ are each equal to 120° one may observe that points $P$ and $Q$ whose distances to the sides of the triangle $ABC$ are inversely proportional have the property that the pedal triangle of one is similar to the antipedal triangle of the other. The antipedal triangle of $P$ being equilateral, the pedal triangle of $Q$ must be equilateral and so $Q$ is given as one of the two intersections of the three Apollonian circles of the triangle $ABC$. (There are two solutions of the Torricelli-Fermat problem since the radical occurring in the distance formula forces an algebraic treatment to consider the problem of stationary values of $PA \pm PB \pm PC$.) The points $P$ and $Q$ are known as isogonal conjugates or as focal conjugates since they are the foci of a conic inscribed in the triangle $ABC$. The argument holds precisely as above in space. Points $P$ and $Q$ whose distances to the faces of the tetrahedron $ABCD$ are inversely proportional have the property that the pedal tetrahedron of one with respect to $ABCD$ is similar to the antipedal tetrahedron of the other. If the areas of the faces of the antipedal tetrahedron of $P$ are equal the sum of the distances $PA$, $PB$, $PC$, $PD$ is a minimum (assuming $P$ inside the antipedal tetrahedron). $Q$ is then such that its pedal tetrahedron is equifacial and on expressing the fact that the opposite edges of an equifacial tetrahedron are equal we find three quadric surfaces of the type $y_2^2 + y_3^2 + 2c_{23}y_2y_3 = y_1^2 + y_4^2 + 2c_{14}y_1y_4$ through $Q$. Here $(y_1, y_2, y_3, y_4)$ are the perpendicular distances of the point $Q$ from the faces of $ABCD$ and $c_{12}$ etc. denote the cosines of the dihedral angles of the tetrahedron $ABCD$. There are, then, 8 points $P = (x_1, x_2, x_3, x_4)$, found by setting $x_1y_1 = x_2y_2 = x_3y_3 = x_4y_4$, making $PA \pm PB \pm PC \pm PD$ stationary. Sturm failed to arrive at this result through paying too much attention to the fact that the trihedral angles $PBCD$, $PCDA$, $PDAB$, $PABC$ are congruent. The 8 points $P$ may well be known as the Neuberg points of the tetrahedron $ABCD$ as they were arrived at by him, in an incidental manner, in 1909.[3]

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[1] Journal für Mathematik, vol. 97 (1884), pp. 49–61.

[2] Journal für Mathematik, vol. 143 (1913), pp. 241–249.

[3] Annales de la Société Scientifique de Bruxelles, vol. 33 (1909), pp. 320–333.

This illustration in support of my thesis has for me the charm of elegance and, if one may say so, of uselessness. Here is a beautiful result unspoiled by any suspicion of a commercial or practical application; just like a beautiful lake surrounded by woods before the engineer comes to dam it or the lumberjack to convert beauty into usefulness. But now I come to an example which will no doubt appeal to many as more forceful since the practical applications of the theorem in question are numerous and of fundamental importance. I refer to the Heaviside expansion theorem which is well known to every research engineer engaged in a study of the problems of telegraphy and telephony. Some years ago the American Mathematical Society had the pleasure of hearing a lecture on the Heaviside operational calculus by Mr. J. R. Carson. This lecture appeared in the Bulletin of the Society for 1926 and those interested in the theorem and its applications are referred to this paper for details. Heaviside stated his theorem in 1886 but his exposition was certainly far from clear and later writers endeavored to give satisfactory proofs. In particular I may refer to a paper by Bromwich in the Proceedings of the London Mathematical Society for 1916.[1] Having occasion to use the theorem during the past year in a problem in electrical engineering, I was rather surprised to find that it is but a very slight modification of a classical result of Cauchy's concerning the determination of a certain particular integral of a linear, non-homogeneous, differential equation with constant coefficients. Heaviside was a genius, original in his methods and rather given to railing at "orthodox" mathematicians for failing to recognize his results or even to be interested in them. I hope to make it clear that a sufficient reply by a mathematician, if one sufficiently addicted to controversy could have been found, would have been the sentence of three words "Read your Cauchy!" Much better than this, however, would have been a clear and simple exposition of Cauchy's theorem. Even today the elementary text books on differential equations seem to ignore Cauchy's result and it is not, then, entirely surprising that Heaviside should have been in ignorance of it or of its connection with his own problem. I was, however, more surprised that Bromwich (for whom, as a member of the famous Galway galaxy, I have naturally a great respect) should have failed to notice this connection.

What, then, is the Heaviside Expansion Theorem and what is the result of Cauchy from which it is at once derivable? In his studies on the propagation of electric vibrations Heaviside was led to a consideration of a system of $n$ ordinary linear differential equations of the type

$$\sum_{\alpha=1}^{n} g_{r\alpha} x_\alpha = f_r(t) \; ; \qquad r = 1,2, \cdots ,n.$$

[1] For a list of references see a paper by M. S. Vallarta, Transactions of the American Institute of Electrical Engineers, vol. 45 (1926), pp. 383–387.

Here the coefficients $g_{rs}$ are usually of the type $g_{rs}=a_{rs}D^2+b_{rs}D+c_{rs}$, where the $a_{rs}$, $b_{rs}$, $c_{rs}$ are all constants and $D$ denotes differentiation with respect to the single independent variable $t$. The $f$'s are given functions of $t$ and it is desired to find functions $(x_1, x_2, \cdots x_n)$ of $t$ which satisfy the differential equations just given and vanish together with their derivatives with respect to $t$ when $t=0$. Following the usual procedure we eliminate all the unknown functions but one, $x_1$, say, by multiplying the equations by the cofactors $G_{r1}$ of the elements of the first column of the determinant of the coefficients and then adding the equations together. This gives an equation of the form $F(D)x_1 = \sum_{\alpha=1}^{n} G_{\alpha 1} f_\alpha(t)$ where $F(D)$ is the determinant of the coefficients $g_{rs}$ and is, in general, a polynomial of order $2n$ in $D$. We may, without lack of generality, assume the coefficient of the highest power of $D$, namely $D^{2n}$ to be unity. On account of the linear character of the differential equations and the special nature of the initial conditions, it is sufficient to consider separately the various equations obtained by taking, one at a time, the terms on the right hand side and we can, at the end, add the solutions of these equations. One of the equations to which we are led in this way is $F(D)x_1=G_{11}f_1(t)$. In order to find a solution of this equation it is convenient to solve the auxiliary equation $F(D)u_1 = f_1(t)$ with the initial conditions to the effect that $u$ and all its derivatives up to the $(2n-1)$st, inclusive, vanish when $t=0$. Then, on account of the constant character of the coefficients of the polynomials $F(D)$ and $G_{11}(D)$, it is apparent that the function $x_1$, defined by $x_1=G_{11}(D)u_1$ will satisfy the equation $F(D)x_1=G_{11}(D)f_1(t)$, for the differential operators $F$ and $G_{11}$ are permutable. Moreover, since $G_{11}$ is a polynomial of degree $(2n-2)$ at most in $D$, it is true that $x_1$ will vanish together with its first derivative when $t=0$. The whole problem, therefore, consists in the determination of that particular solution of an equation of the type $F(D)u=f(t)$ which vanishes, together with its derivatives up to the $(2n-1)$st, inclusive, when $t=0$. Now this is the question which was solved by Cauchy (the essentials of his solution going back, indeed, to Euler) and of which the solution may be said to have been classical at least a half-century before Heaviside became interested in his problem. It is, I think, hard to see at just what point in the preceding page of argument anyone familiar with the classical solution would fail to observe the immediate connection with the Heaviside question. But the unfortunate state of affairs is that the elementary texts on the theory of differential equations with which I am familiar do not explain the Cauchy method and the situation must have been the same some forty years ago when Heaviside was seeking eagerly for help from the mathematicians.[1]

[1] Following the delivery of this address I had the pleasure of a conversation with Dr. T. C. Fry of the Bell Telephone Laboratories in the course of which he informed me that after a survey of existing

What is Cauchy's method? In order to find that particular solution of a non-homogeneous linear differential equation of order $m$ which vanishes together with its derivatives up to the $m-1$st, inclusive, when $t=0$, he proceeds as follows. Writing the equation to be solved as $F(D)x=f(t)$, where the coefficient of $D^m$ in $F(D)$ is taken as unity, the general solution of the corresponding homogeneous equation $F(D)u=0$ is of the type $u=C_1u_1+C_2u_2+\cdots+C_mu_m$, the $C$'s being arbitrary constants and the $u$'s a set of $m$ distinct solutions of the homogeneous equation. These constants may be so determined that $u$ and its derivatives up to the $(m-2)$nd, inclusive, vanish when $t=\tau$, the $(m-1)$st derivative taking at that instant the value $f(\tau)$; here $\tau$ is a fixed but arbitrary value of $t$. The solution $u$ of the homogeneous equation which is found in this way involves the parameter $\tau$ and, to stress this, we may write it $u(t,\tau)$. Then Cauchy showed that the function $x$ defined by $x=\int_0^t u(t,\tau)d\tau$ is that particular solution of the non-homogeneous equation $F(D)x=f(t)$ which vanishes with all its derivatives up to the $(m-1)$st, inclusive, when $t=0$. In fact since $u(t,\tau)$ and its derivatives up to the $(m-2)$nd, inclusive, vanish when the two arguments $t$ and $\tau$ become equal we have

$$Dx = \int_0^t Du(t,\tau)d\tau\ ;\cdots D^{n-1}x = \int_0^t D^{n-1}u(t,\tau)d\tau$$

whilst

$$D^nx = \int_0^t D^nu(t,\tau)d\tau + D^{n-1}u(t,\tau)_{\tau=t} = \int_0^t D^nu(t,\tau)d\tau + f(t),$$

so that

$$F(D)x = \int_0^t F(D)u(t,\tau)d\tau + f(t) = f(t).$$

In the particular case when the coefficients of $F(D)$ are constants and its zeros simple, the explicit expression for $x$ is easy to find. Writing $F(D)=(D-r_1)(D-r_2)\cdots(D-r_m)$ we may set

$$u_1 = C_1e^{r_1(t-\tau)}\ ;\cdots\ ;u_m = C_me^{r_m(t-\tau)}$$

and we have to determine the constants $C$ so as to satisfy the equations

$$C_1+C_2+\cdots+C_m=0\ ;\qquad r_1C_1+r_2C_2+\cdots+r_mC_m=0\ ;\text{etc.},$$

the last equation being

$$r_1^{m-1}C_1+r_2^{m-1}C_2+\cdots+r_m^{m-1}C_m=f(\tau).$$

These equations show that the $C$'s are the numerators in the development of $f(\tau)/F(r)$ into its simple fractions; that is, we have the identity $f(\tau)/F(r)$

books they have had to mimeograph a special course in differential equations to meet their needs. Incidentally the deduction of Heaviside's expansion theorem from Cauchy's method, in much the same manner as that indicated here, is given, I am informed, in that course.

$=C_1/(r-r_1)+\cdots+C_m/(r-r_m)$. In fact if we expand each of the simple fractions on the right near $r=\infty$ and express that $r=\infty$ is a zero of order $m$ of the left hand side, the first term in its expansion being $f(\tau)/r^m$, we get the equations just given for the $C$'s. The observation that the points $(r_1, r_2, \cdots r_m)$ are simple poles of the expression on the left gives at once, on expanding near one of these poles, that $C_p=f(\tau)/F'(r_p)$, and so

$$u(t,\tau) = f(\tau) \sum_1^m e^{r_p(t-\tau)}/F'(r_p).$$

In the particular case when $f(t)$ is constant and equal to unity, say, we have

$$x(t) = \int_0^t u(t,\tau)d\tau = \sum_1^m \frac{e^{r_p t}}{r_p F'(r_p)} - \sum_1^m \frac{1}{r_p F'(r_p)}.$$

The constant term on the right is $1/F(0)$ since we have the identical relation

$$1/F(r) = \sum_{p=1}^m C_p/(r-r_p) = \sum_{p=1}^m 1/[(r-r_p)F'(r_p)],$$

so that we have

$$x(t) = \frac{1}{F(0)} + \sum_{p=1}^m \frac{e^{r_p t}}{r_p F'(r_p)}.$$

In order to step from this classical result to the Heaviside expansion theorem, all we have to do is to operate on $x(t)$ with the operator $G_{11}$ and we get

$$h(t) = G_{11}x(t) = \frac{G_{11}(0)}{F(0)} + \sum_{p=1}^m \frac{G_{11}(r_p)e^{r_p t}}{r_p F'(r_p)}$$

which is the expansion theorem in question. It may be appropriately remarked here that the contour integral method adopted by Bromwich and Wagner connects with the method explained here through the evaluation of the $C$'s as the residues of the function $1/F(r)$ at its simple poles $r_p$.[1]

It seems clear from these examples that carefulness in exposition is necessary even when writing for experts. The slightest deviation from the natural course of the argument often suffices to make quite difficult what would otherwise be simple. But this is not the worst result. A situation sometimes develops where a man like Heaviside, by some flash of intuition, obtains a result whose derivation is not very clear even to the discoverer. Yet the result is true and a kind of "operational mathematics" develops. Amongst the followers of the new creed a certain laxity of mathematical morality may be observed. We are

[1] The case where the zeros of $F(r)$ are not all simple is dealt with in an article by the writer in the Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 81–89.

told by them that the method "works"; that they do not know how it works and in extreme cases we are given to understand that they do not particularly care. Those of you who had the pleasure of listening two days ago to Professor Swann's address on the "New Quantum Mechanics" must feel that a state of affairs similar to that which arose during the last decade of the nineteenth century in regard to Heaviside's operational calculus has now arisen in Physics. Let us hope that some one will soon be able to give a logically sound and connected exposition of the new theories which can be readily understood.

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## THE ROTATIONAL DERIVATIVE AND SOME APPLICATIONS[1]

By M. E. MULLINGS, University of Texas

**1. Introduction.** The problem of the rotation, and the description of the motion of a rotating body has been considered from various angles by the use of vector analysis in the component form. But the practice of referring problems in vector analysis back to component forms loses many of the advantages to be gained by the vector methods. In 1921 Gans[2] gave a definition, from a purely geometric point of view, of a component of the curl of a vector function and by combining the components in three fundamental directions obtained the curl. He did not state the conditions under which the curl would exist, and in his proof of Stokes' Theorem based on this geometric definition he does not state the conditions, nor does he carry his proof through in detail. The notion of a directional derivative and some of its applications is quite familiar, even to students of advanced calculus. I propose, in this paper, to give a definition of a similar nature, which I shall call the rotational derivative about an axis, and to use it to prove several theorems based on carefully formulated conditions.

I shall use the Greek letter $\rho$ throughout this paper to denote the position vector. The directional derivative of a vector function $\phi(\rho)$ about a direction $\alpha_1$ may be considered as a derivative with respect to the unit vector $\alpha_1$, and I shall use the notation $D_{\alpha_1}\phi$ to designate the directional derivative of $\phi$.

**2. Definitions.** (a). By a curve of type $C$ I shall mean a curve given by:

(1) $\rho = C(s)$ *where* $s$ *is the length of arc.*

(2) $C(s)$ *has a continuously turning tangent except for a linear null set of points.*

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[1] This paper is a portion of a thesis for the M. A. degree presented to the University of Texas in June, 1926, which I prepared, independently of the work of Gans, under the direction of Dr. H. J. Ettlinger.

[2] *Einführung in die Vektoranalysis mit Anwendungen auf die Mathematische Physik* (Teubner, Leipzig-Berlin, 1921), Ch. II pp. 34–38.

(3) *$C(s)$ has a backward and a forward tangent everywhere.*

(b) A bifacial surface $S$ defined by the equation $\rho = S(u,v)$ is said to be a simple surface if:

(1) *$S(u,v)$ is continuous in $u$ and $v$, where $u$ is the element of length along the curve $\rho = S(u,\bar{v})$ and $v$ along the curve $\rho = S(\bar{u},v)$, $\bar{u}$ and $\bar{v}$ being fixed values of $u$ and $v$.*

(2) *$S_{\alpha_1}(u,v) = D_{\alpha_1}S$ and $S_{\beta_1}(u,v) = D_{\beta_1}S$, where $\alpha_1$ is a unit vector tangent to the curve $\rho = S(u,\bar{v})$ and $\beta_1$ is a unit vector tangent to the curve $\rho = S(\bar{u},v)$.*

(3) *$S$ has a normal $N(u,v) = S_{\alpha_1}(u,v) \times S_{\beta_1}(u,v)$ existing everywhere except for a two-dimensional null set and continuous where it exists.*

(4) *At any point $P$ at which the normal does not exist, there is a manner of approach to $P$ such that there is a limiting normal at $P$, i. e., a vector $\overline{N}$ such that*

$$\lim_{P' = P} N(u,v) = \overline{N}.$$

(c). *Rotational Derivative about an Axis.* In a vector field $\phi(\rho)$, pass a plane with normal direction $\alpha_1$ through a point $P$, and let $\Delta a$ represent the magnitude of the area of a portion of the plane bounded by a closed curve $L$ of type $C$, and having $P$ as an interior point. *Then if*

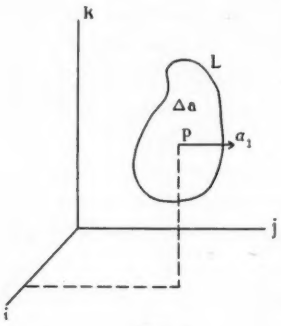


FIG. 1.

$$\lim_{\Delta a = 0} \frac{\int_L \phi(\rho) \cdot d\rho}{\Delta a} \alpha_1$$

*exists, where the integral is taken around $L$ in the positive direction and $\Delta a$ approaches zero in such a manner that the maximum dimension of $\Delta a$ approaches zero, we will call that limit the rotational derivative $\phi_{\alpha_1}$ of the function $\phi(\rho)$ about the direction $\alpha_1$ at the point $P$.*

This definition will hold equally well if the plane through $P$ is replaced by a simple surface having a normal at $P$ whose direction is $\alpha_1$.

**3.** With the above definitions as a basis and making use of the ordinary operations of the vector analysis I shall now proceed to establish three theorems.

THEOREM I. *Hypothesis:*

(1). *$\phi(\rho) = Xi + Yj + Zk$ is a vector function in the neighborhood of a point $P_0$.*

(2). *$Z_y$ and $Y_z$ exist and are continuous in $\rho$ in the neighborhood of $P_0$.*

(3). *$\pi$ is a plane parallel to the $jk$-plane at a distance $p$ from the point $P_0$.*

(4). *$P$ is the projection of $P_0$ on $\pi$.*

(5). *$L$ is a closed curve of type C with $P$ in its interior, lying in $\pi$, and enclosing an area $\Delta a$.*

(6). *The maximum dimension of $\Delta a$ approaches zero as the distance $p$ approaches zero.*

*Conclusion:* $$\lim_{\Delta a=0} \frac{\int_L \phi(\rho)\cdot d\rho}{\Delta a} i = [Z_y(\rho_0) - Y_z(\rho_0)]\, i$$

PROOF: Let $d\rho = dx\, i + dy\, j + dz\, k$. Then since the curve $L$ is in a plane parallel to the jk-plane we have that $dx=0$; hence $\int_L \phi(\rho)\cdot d\rho = \int_L (Ydy+Zdz)$ and by Green's theorem[1] for two dimensions we have

$$\int_L (Ydy + Zdz) = \iint_{\Delta a} (Z_y - Y_z)dydz.$$

Now applying the law of the mean to the second member, we have

$$\iint_{\Delta a}(Z_y - Y_z)dydz = (\bar{Z}_y - \bar{Y}_z)\iint_{\Delta a} dydz$$

where $\bar{Z}_y$ and $\bar{Y}_z$ are the values of $Z_y$ and $Y_z$ at some point $\bar{P}$ in $\Delta a$. But

$$\iint_{\Delta a} dydz = \Delta a\,.$$

Therefore

$$\lim_{\Delta a=0} \frac{\int_L \phi(\rho)\cdot d\rho}{\Delta a} i = \lim_{\Delta a=0} \frac{(\bar{Z}_y - \bar{Y}_z)\Delta a}{\Delta a} i = [Z_y(\rho_0) - Y_z(\rho_0)]\, i\,.$$

THEOREM II. *Hypothesis:*

(1). *$\phi(\rho) = Xi + Yj + Zk$ is a vector function in the neighborhood of $P_0$.*

(2). *$X_y$, $X_z$, $Y_z$, $Y_z$, $Z_z$, and $Z_y$ exist and are continuous in $\rho$ in the neighborhood of $P_0$.*

(3). *$\phi(\rho)$ has a rotational derivative at $P_0$ for every $\alpha_1$.*

*Conclusion: There exists a vector curl $\phi = \phi_i + \phi_j + \phi_k$ such that curl $\phi = \nabla \times \phi$ and $\phi_{\alpha_1} = (\alpha_1 \cdot curl\, \phi)\alpha_1$.*

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[1] W. F. Osgood, *Advanced Calculus* (The MacMillan Co., New York, 1925), pp. 222–224.

PROOF: Through $P_0$ pass a plane with normal direction $\alpha_1$. At a distance $\rho$ from $P_0$ along $\alpha_1$ choose a point $P$ and pass planes through it parallel to the fundamental planes. In the resulting tetrahedron $P$-$ABC$ let $\Delta a$ be the magnitude of the area of the base $ABC$, and let $L$ be the curve bounding $\Delta a$, let $\Delta a_1$, $\Delta a_2$, and $\Delta a_3$ be the magnitudes of the areas of the faces $PBC$, $PCA$, and $PAB$ respectively, and $L_1$, $L_2$, and $L_3$ the bounding curves. Then

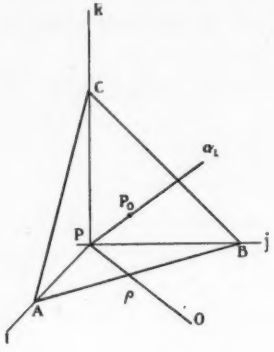


FIG. 2.

$$\phi_{\alpha_1} = \operatorname{Lim}_{\Delta a=0} \frac{\alpha_1 \int_L \phi(\rho)\cdot d\rho}{\Delta a}$$

$$= \alpha_1 \left[ \operatorname{Lim}_{\Delta a_1=0} \frac{\int_{L_1} \phi(\rho)\cdot d\rho}{\Delta a_1} \frac{\Delta a_1}{\Delta a} \right.$$

$$\left. + \operatorname{Lim}_{\Delta a_2=0} \frac{\int_{L_2} \phi(\rho)\cdot d\rho}{\Delta a_2} \frac{\Delta a_2}{\Delta a} + \operatorname{Lim}_{\Delta a_3=0} \frac{\int_{L_3} \phi(\rho)\cdot d\rho}{\Delta a_3} \frac{\Delta a_3}{\Delta a} \right].$$

But

$$\frac{\Delta a_1}{\Delta a} = \alpha_1 \cdot i, \qquad \frac{\Delta a_2}{\Delta a} = \alpha_1 \cdot j, \quad \text{and} \quad \frac{\Delta a_3}{\Delta a} = \alpha_1 \cdot k,$$

and as $\rho$ approaches zero, each of the three faces satisfies the conditions of Theorem I and we have

$$\operatorname{Lim}_{\Delta a_1=0} \frac{\int_{L_1} \phi(\rho)\cdot d\rho}{\Delta a_1} = Z_y(\rho_0) - Y_z(\rho_0) \text{ etc.}$$

Hence

$$\phi_{\alpha_1} = \alpha_1[(Z_y - Y_z)\alpha_1 \cdot i + (X_z - Z_z)\alpha_1 \cdot j + (Y_z - X_y)\alpha_1 \cdot k].$$

Now project $\Delta a_1$ into $\Delta a_1'$ on a plane through $P_0$ parallel to the jk-plane. Then by definition (c) we have

$$\phi_i = \operatorname{Lim}_{\Delta a_1=0} \frac{\int_{L_1'} \phi(\rho)\cdot d\rho}{\Delta a_1'} i$$

and by Theorem I we have $\phi_i = (Z_y - Y_z)i$.
Similarly $\phi_j = (X_z - Z_z)j$, and $\phi_k = (Y_z - X_y)k$.

Substituting the above, we have the second part of our conclusion as follows,

$$\phi_a = [\phi_i \cdot \alpha_1 + \phi_j \cdot \alpha_1 + \phi_k \cdot \alpha_1]\alpha_1 = (\alpha_1 \cdot \text{curl } \phi)\alpha_1 .$$

To get the first part we have

$$\text{curl } \phi = \phi_i + \phi_j + \phi_k = (Z_y - Y_z)i + (X_z - Z_x)j + (Y_x - X_y)k = \nabla \times \phi .$$

THEOREM III. The ease with which Stokes' Theorem follows from the above form of definition of curl $\phi$ is most suprising. The theorem in full is:

*Hypothesis:*

(1). *$S$ is a simple surface bounded by a closed curve $L$ of type $C$.*

(2). *$\phi(\rho)$ is a vector function defined on $S$ and $L$.*

(3). *$\int_L \phi(\rho) \cdot d\rho$ exists.*

(4). *Curl $\phi$ exists on $S$ and $L$ and is continuous in $\rho$ everywhere except for a two-dimensional null set on $S$.*

(5). *Curl $\phi$ is bounded on $S$ and $L$.*

*Conclusion:* $\iint_S \text{Curl } \phi \cdot dS = \int_L \phi(\rho) \cdot d\rho$.

PROOF: Divide $S$ into $n$ subdivisions $\Delta a_{in}$ each bounded by a closed curve $L_{in}$ of type $C$. Choose any point $P_{in}$ with coordinate $\rho_{in}$ in the subdivision $\Delta a_{in}$, and let $\alpha_{in}$ be the normal, or a limiting normal, at $P_{in}$. Consider

$$h_{in} = \frac{\int_{L_{in}} \phi(\rho) \cdot d\rho}{\Delta a_{in}}$$

and

$$H_{in} = \text{curl } \phi(\rho_{in}) \cdot \alpha_{in}) .$$

where the integral is taken around $L_{in}$ in a positive direction. Now $|H_{in}| < M$ by hypothesis, and since

$$\text{curl } \phi \cdot \alpha_{in} = \lim_{\Delta a_{in} = 0} \frac{\int_{L_{in}} \phi(\rho) \cdot d\rho}{\Delta a_{in}}$$

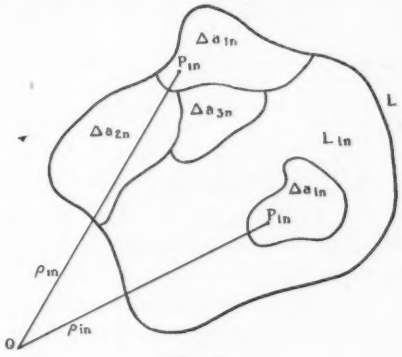


FIG. 3.

we have $|h_{in}| < M$. The difference of two bounded functions is bounded; hence $|h_{in} - H_{in}| < 2M$ for every $i$ and $n$.

For a fixed point $P_0$ on $S$ at which curl $\phi$ is continuous and $\alpha_0$ exists, let $\Delta a_0$ be the subdivision containing $P_0$ for every value of $n$. Let $L_0$ be the curve bounding $\Delta a_0$, and let $h_0$ and $H_0$ be the values of $h_{in}$ and $H_{in}$ associated with $\Delta a_0$. If $n$ increases without limit in such a manner that the maximum dimension of the maximum $\Delta a_{in}$ approaches zero,

$\operatorname{Lim}_{n=\infty} H_0 = \operatorname{curl} \phi(\rho_0) \cdot \alpha_0$ and $\operatorname{Lim}_{n=\infty} h_0 = \operatorname{curl} \phi(\rho_0) \cdot \alpha_0$. Hence $\operatorname{Lim}_{n=\infty} (h_0 - H_0) = 0$.

Form the sum $$\sum_1^n h_{in}\Delta a_{in} = \sum_1^n \int_{L_{in}} \phi(\rho) \cdot d\rho .$$

We note that the portion of the totality of the curves $L_{in}$ that is not a part of $L$ is traversed twice, once in each direction; hence the net result of integrating over that portion is zero, and the other portion is such that the boundary $L$ is traversed once in the positive direction; hence

$$\sum_1^n \int_{L_{in}} \phi(\rho) \cdot d\rho = \int_L \phi(\rho) \cdot d\rho \text{ and then } \operatorname{Lim}_{n=\infty} \sum_1^n h_{in}\Delta a_{in} = \int_L \phi(\rho) \cdot d\rho .$$

If we identify the surface $S$ plus the boundary $L$ as the set $E$ and the subdivisions $\Delta a_{in}$ as the measure $e_{in}$ of the sub-sets $E_{in}$, the hypotheses of the Duhamel-Moore theorem[1] are satisfied. Hence

$$\lim_{n=\infty} \sum_1^n H_{in}\Delta a_{in} = \int_L \phi(\rho) \cdot d\rho .$$

Replacing $\alpha_{in}\Delta a_{in}$ by the vector quantity $\Delta S_{in}$,

$$\lim_{n=\infty} \sum_1^n H_{in}\Delta a_{in} = \lim_{n=\infty} \sum_1^n \operatorname{curl} \phi(\rho_{in}) \cdot \Delta S_{in} = \iint_S \operatorname{curl} \phi \cdot dS .$$

Hence

$$\iint_S \operatorname{curl} \phi \cdot dS = \int_L \phi(\rho) \cdot d\rho .$$

4. It is interesting to note that the above proof of Stokes' theorem is very similar to that given by Gans. However his proof is scarcely more than an outline, as he does not attempt to justify the steps nor to give all the conditions necessary.

It is surprising that in such standard works as Gibbs'[2] text book on vector analysis a statement should be found to the effect that a definition of the nature of the one given herein is of more theoretical than practical interest, in view of the fact that the proof of Stokes' theorem given therein is considerably longer and more difficult to understand than the above. Furthermore, it is very loosely written, depending more on intuition than carefully formulated

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[1] R. L. Moore, *On Duhamel's theorem*, Annals of Mathematics, (2), vol. 13(1912), p. 161.
H. J. Ettlinger, *A simple form of Duhamel's theorem* and some applications, this MONTHLY, vol. XXIX (1922), p. 239.

[2] *Vector Analysis*, Yale University Press (New Haven, 1922), p. 194. See pp. 184-193 for proof of Stokes' theorem.

arguments. Osgood[1] and Woods[2] give proofs of Stokes' theorem which are more rigorous than that of Gibbs, Osgood's being more rigorous than that of Woods, but in each the arguments are no more elementary in nature than those in the above proof. Moreover, the kind of bounding curve used is not stated in either of the proofs, and Woods does not state what kind of functions he is using, nor in any way describe the surface used. Osgood is not exact in his statement concerning continuity of his functions, and requires a bilateral surface having a continuous normal everywhere, which condition is less general than the case that I have considered.

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## A CURIOUS CASE OF THE USE OF MATHEMATICAL INDUCTION IN GEOMETRY

By J. V. USPENSKY, Carleton College

It is known that the ratio $\sin x/x$ continually decreases when $x$ increases from 0 to $\pi$. Ordinarily the proof is given by means of trigonometry or calculus. The purpose of this note is to show how an equivalent statement can be established by purely geometric means using the process of reasoning by induction, which certainly is a rare case in geometry. Following is the fundamental theorem:

THEOREM I. *If $\alpha$ and $\beta$ denote the angles at the base $AB$ of a triangle $ABC$ (Fig. 1) opposite to the sides $CB$ and $CA$, then the following inequalities hold*

$$\frac{CA}{BA} > \frac{\beta}{\alpha+\beta} \text{ and } \frac{CB}{AB} > \frac{\alpha}{\alpha+\beta}.$$

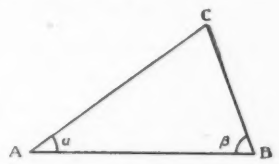


FIG. 1.

PROOF: (a) First let us consider triangles with commensurable angles at the base, so that we can set

$$\alpha = p\delta \text{ and } \beta = q\delta$$

where $p$ and $q$ are relatively prime integers and $\delta$ is a certain angle. Supposing that the above inequalities have already been proved in every case when the corresponding sum $p+q$ is *less* than a given integer $N>2$, we shall show that they continue to hold when $p+q=N$. Let $ABC$ be a given triangle (Fig. 2) with the angles at the base $p\delta$ and $q\delta$, where $p$ and $q$ are relatively prime integers and $p+q=N>2$. The sides $AC$ and $CB$ cannot be equal, and we

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[1] Loc. cit. pp. 237–241.

[2] *Advanced Calculus* (Ginn and Co., Boston, 1926), pp. 197–200.

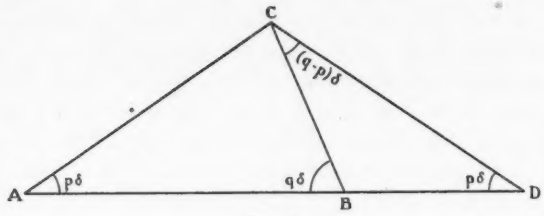


FIG. 2.

can suppose $AC>BC$. Describe from $C$ as center and with $CA$ as radius the circumference which passes through $A$ and cuts the prolonged base $AB$ at the point $D$. In the triangle $ACD$,

(1) $$AC=CD \text{ and } 2AC>AD=AB+BD.$$

Considering $CD$ as the base of the triangle $CBD$ we find that the adjoining angles are $p'\delta$, $q'\delta$, where $p'=p$ and $q'=q-p$, so that

$$p'+q'=q<N.$$

As the sum $p'+q'$ is *less* than $N$ we have by supposition

(2) $$BD>\frac{q-p}{q}CD \quad \text{or} \quad BD>\frac{q-p}{q}AC.$$

Adding (1) and (2) we get

$$2AC+BD>\frac{q-p}{q}AC+AB+BD,$$

whence

$$\frac{q+p}{q}AC>AB \text{ or } \frac{AC}{AB}>\frac{q}{p+q}=\frac{\beta}{\alpha+\beta}.$$

From the same triangle $CBD$ we get

$$CB>\frac{p}{q}CA \text{ and } CB>\frac{p}{p+q}AB \text{ or } \frac{CB}{AB}>\frac{\alpha}{\alpha+\beta}.$$

Now if the sum $p+q$ has the least possible value, namely 2, it must be that $p=q=1$; that is the triangle is isosceles, and for such a triangle it is obvious that

$$\frac{AC}{AB}>\frac{1}{2} \text{ and } \frac{BC}{AB}>\frac{1}{2},$$

so that our inequalities hold true when $N=2$. Then they will continue to hold for $N=3, 4, 5, \cdots$, that is in every possible case, provided the angles $\alpha$ and $\beta$ are commensurable.

(b) Next suppose $\alpha$ and $\beta$ incommensurable. Divide $\alpha$ into $p$ equal parts, so that $\alpha=p\delta$ and determine an integer $q$ by the condition $q\delta<\beta<(q+1)\delta$. Then construct the triangle $ABC'$ with the angles $p\delta$ and $q\delta$ at its base $AB$. As these angles are commensurable we have

$$\frac{AC'}{AB} > \frac{q}{p+q} \text{ and } \frac{BC'}{AB} > \frac{p}{p+q}.$$

Now if $p$ increases indefinitely, $AC'$ and $BC'$ approach respectively to $AC$ and $BC$, and at the same time the ratios $q/(p+q)$ and $p/(p+q)$ converge to the limits $\beta/(\alpha+\beta)$ and $\alpha/(\alpha+\beta)$. Performing the passage to the limit we conclude from the preceding inequalities that

$$\frac{AC}{AB} \geqq \frac{\beta}{\alpha+\beta} \text{ and } \frac{BC}{AB} \geqq \frac{\alpha}{\alpha+\beta}.$$

It is easy to show that the sign $=$ is excluded. Repeat the same construction as in Fig. 1. From the triangle $BCD$ we get

$$BD \geqq \frac{\beta-\alpha}{\beta} AC, \text{ and furthermore } 2AC > AB + BD,$$

whence $$\left(2 - \frac{\beta-\alpha}{\beta}\right) AC > AB, \text{ or } \frac{AC}{AB} > \frac{\beta}{\alpha+\beta}.$$

Combining this inequality with $\frac{CB}{AC} \geqq \frac{\alpha}{\beta}$, which follows from the same triangle $BCD$, we get finally $\frac{CB}{AB} > \frac{\alpha}{\alpha+\beta}$. *Q.E.D*

THEOREM II. *Two arcs $AB$ and $AC$, neither exceeding a semicircumference, being taken on the same circle and the latter being the greater of the two, the following inequality holds:*

$$\frac{AB}{AC} > \frac{\text{arc}AB}{\text{arc}AC}.$$

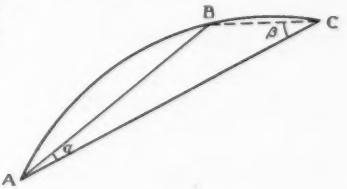


FIG. 3.

PROOF: Applying the preceding theorem to the triangle $ABC$ we have

$$\frac{AB}{AC} > \frac{\beta}{\alpha+\beta}.$$

On the other hand $$\frac{\text{arc}\,AB}{\text{arc}\,BC} = \frac{\beta}{\alpha},$$

whence $$\frac{\text{arc}\,AB}{\text{arc}\,AC} = \frac{\beta}{\alpha+\beta} \text{ and } \frac{AB}{AC} > \frac{\text{arc}\,AB}{\text{arc}\,AC}.$$ *Q.E.D.*

COROLLARY I. Taking a certain unit of length to measure distances we can express the lengths of the chords $AB$ and $AC$ by the *numbers* $s$ and $s'$. Taking a certain arc of our circle, e. g. the whole circumference, as a unit we can express the measures of the arcs $AB$ and $AC$ by two *numbers* $\sigma$ and $\sigma'$. We have then the following relation:

$$\frac{s}{s'} > \frac{\sigma}{\sigma'} \quad \text{or} \quad \frac{s}{\sigma} > \frac{s'}{\sigma'}.$$

if $\sigma' > \sigma$, *and this implies that* $\sin x/x$ *diminishes when* $x$ *increases from* 0 *to* $\pi/2$.

THEOREM III. *Denote by* $P$ *and* $P'$ *the perimeters of two polygons inscribed in the same circle. If the greatest side of the second is less than the smallest side of the first, then*

$$P' > P.$$

PROOF: Denote by $s_1, s_2, s_3, \cdots, s_n$ the measures of the sides of the first polygon in the *increasing* order of magnitude, and by $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$ the measures of the subtended arcs, the whole circumference being taken as a unit. Denote by $s_1', s_2', s_3', \cdots, s_m'$ the measures of the sides of the second polygon in the *decreasing* order of magnitude, $\sigma_1', \sigma_2', \sigma_3', \cdots, \sigma_m'$ being the measures of the subtended arcs. As

$$\sigma_1 < \sigma_2 < \sigma_3 < \cdots < \sigma_n \quad \text{and} \quad \sigma_1' > \sigma_2' > \sigma_3' > \cdots > \sigma_m',$$

we have by the preceding corollary

$$\frac{s_1}{\sigma_1} > \frac{s_2}{\sigma_2} > \frac{s_3}{\sigma_3} > \cdots > \frac{s_n}{\sigma_n} \text{ and } \frac{s_1'}{\sigma_1'} < \frac{s_2'}{\sigma_2'} < \frac{s_3'}{\sigma_3'} < \cdots < \frac{s_m'}{\sigma_m'}$$

whence it follows that

$$\frac{s_1' + s_2' + \cdots + s_m'}{\sigma_1' + \sigma_2' + \cdots + \sigma_m'} > \frac{s_1'}{\sigma_1'} \text{ and } \frac{s_1 + s_2 + \cdots + s_n}{\sigma_1 + \sigma_2 + \cdots + \sigma_n} < \frac{s_1}{\sigma_1}.$$

By supposition $\sigma_1' < \sigma_1$, whence by the same corollary $\dfrac{s_1}{\sigma_1} < \dfrac{s_1'}{\sigma_1'}$.

That is $$\frac{s_1 + s_2 + \cdots + s_n}{\sigma_1 + \sigma_2 + \cdots + \sigma_n} < \frac{s_1' + s_2' + \cdots + s_m'}{\sigma_1' + \sigma_2' + \cdots + \sigma_m'}.$$

Now $$\sigma_1 + \sigma_2 + \cdots + \sigma_n = 1 \text{ and } \sigma_1' + \sigma_2' + \cdots + \sigma_m' = 1$$

and consequently $$s_1 + s_2 + \cdots + s_n < s_1' + s_2' + \cdots + s_m'.$$

That is $$P < P'. \qquad Q.E.D.$$

COROLLARY. Perimeters of the regular polygons inscribed in the same circle increase with the increasing number of sides.

# NOTE ON THE SMOOTHING OF CURVES

By W. E. MILNE and V. ROJANSKY, University of Oregon

In many statistical investigations it is found that the values of a function obtained from observation are quite irregular, and it is assumed that these irregularities are due to accidental errors or deviations from the unknown "true values" of the function. Sometimes it is desirable to remove as far as possible these fluctuations, and to obtain new values which are more regular and presumably nearer on the average to the "true values" of the function. For this purpose either free-hand graphical smoothing or some type of moving average is recommended in text-books on statistics.

The object of this note is to suggest a method of smoothing which has a plausible justification in theory, and which in practice yields quite satisfactory results.

For the purpose in hand we shall say that a set of values of a function at equally spaced values of the argument is *regular* if their differences of a certain order, say $m$, are negligible. For simplicity we shall take $m=4$ in the sequel, though the reasoning applies equally well for any even value of $m$.

Let $y_0, y_1, \cdots, y_n$ be the values obtained for a function $f(x)$ at equally spaced values of $x$, let $u_0, u_1, \cdots, u_n$ be the corresponding "true values" of $f(x)$, and let $e_0, e_1, \cdots, e_n$ be the corresponding errors, so that

$$y_i = u_i + e_i. \tag{1}$$

We shall assume:

A. That the values $u_i$ are *regular*.

B. That the errors $e_i$ are due purely to chance, so that positive and negative values are equally probable.

Starting from these two assumptions we wish to determine a formula for smoothing the $y_i$ which shall have the following two properties:

I. The formula will leave the $y_i$ unchanged if they are already regular.

II. If they are not regular it will give new values $y_i'$ for which the new errors $e_i'$ satisfy the inequality

$$\sum(e_i')^2 < \sum e_i^2 .$$

Now the first condition will be satisfied by a formula of the type

$$y_i' = y_i + k\Delta^4 y_i \quad (i=2, 3, \cdots, n-2), \tag{2}$$

in which $k$ is arbitrary and where $\Delta^4 y_i = y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}$. By substitution from (1) equation (2) becomes $y_i' = u_i + e_i + k\Delta^4 e_i$, since by assumption A it follows that $\Delta^4 u_i = 0$. Therefore

$$e_i' = e_i + k\Delta^4 e_i, \text{ and } \Sigma(e_i')^2 = \Sigma e_i^2 + 2k\Sigma e_i \Delta^4 e_i + k^2 \Sigma(\Delta^4 e_i)^2,$$

the summation extending from $i=2$ to $i=n-2$. The value of $k$ which makes $\Sigma(e_i')^2$ a minimum is

$$k = -\left(\sum e_i \Delta^4 e_i\right) / \left[\sum (\Delta^4 e_i)^2\right], \tag{3}$$

and the minimum itself is

$$\sum (e_i')^2 = \sum e_i^2 - \frac{(\sum e_i \Delta^4 e_i)^2}{\sum (\Delta^4 e_i)^2}. \tag{4}$$

We have assumed in $B$ that positive and negative errors are equally probable so that in general we may neglect the sums of cross products. We shall further assume that to a sufficient degree of approximation

$$\sum_{i=1}^{n-3} e_i^2 = \sum_{i=2}^{n-2} e_i^2, \text{ etc.} \tag{5}$$

If now we substitute in (3) the value of $\Delta^4 e_i$, expand, drop sums of cross products, and apply equations of type (5) to sums of squares we find that (3) reduces to

$$k = -6/70, \tag{6}$$

and similarly (4) becomes

$$\sum (e_i')^2 = (34 \sum e_i^2)/70 \tag{7}$$

so that the sum of the squares of the errors is approximately cut in half. The formula (2) may now be written

$$y_i' = (1/70)[-6y_{i-2} + 24y_{i-1} + 34y_i + 24y_{i+1} - 6y_{i+2}]. \tag{8}$$

This is our desired formula. However for actual practice we can obtain a formula which is nearly as accurate and easier to use by taking $k=-1/12$ instead of $-6/70$. Then in place of (8) we have

$$y_i' = (1/12)[-y_{i-2} + 4y_{i-1} + 6y_i + 4y_{i+1} - y_{i+2}], \tag{9}$$

and $\Sigma(e_i')^2 = 35/72 \Sigma e_i^2$ in place of (7), a result which is for practical purposes just as good as (7).

It remains to consider the four values $y_0'$, $y_1'$, $y_{n-1}'$, $y_n'$, which are not provided for by formula (8) or (9). First however we note that many frequency functions occurring in statistical studies taper off toward zero at both ends, and when this is the case it is merely a matter of notation to extend (8) and (9) beyond the ends so as to cover all significant values. When this can not be done we may by an easy modification of the preceding methods obtain formulas of the type

$$y_i' = (1/70)(4y_{i-1} + 54y_i + 24y_{i+1} - 16y_{i+2} + 4y_{i+3}],$$

and

$$y_i' = (1/70)\,[69y_i + 4y_{i+1} - 6y_{i+2} + 4y_{i+3} - y_{i+4}]$$

which take care of the first two values, and the two formulas symmetrical to these which take care of the last two values.

Formula (8) may be derived by another method which throws additional light on its significance. For if by the method of least squares we fit a cubic to the five values $y_{i-2}$, $y_{i-1}$, $y_i$, $y_{i+1}$, $y_{i+2}$, we find that the value which the cubic gives in place of $y_i$ is precisely the value obtained by (8).

Turning now briefly to the general case where $m$ is any even integer, we notice that the expression for $k$ has in the numerator the mid-coefficient of the binomial expansion of $m$th degree, and in the denominator has the sum of the squares of all the binomial coefficients of $m$th degree. Thus taking account of the fact that the mid-coefficient is $m!/[(m/2)!]^2$ and the sum of the squares of the coefficients is equal to the mid-coefficient in the expansion of degree $2m$, that is $(2m)!/(m!)^2$, we have

$$k = -\frac{(m!)^3}{[(m/2)!]^2(2m!)}. \tag{10}$$

Using this value of $k$ we have as the general result in place (4)

$$\sum(e')^2 = \sum e^2\left[1 - \left\{\frac{m!}{(m/2)!}\right\}^4 \cdot \frac{1}{(2m)!}\right]. \tag{11}$$

Since the right hand side of (11) increases as $m$ increases and approaches $\sum e^2$ as a limit it is apparent that we should use no larger value of $m$ than is necessary to make the $m$th differences of the "true values" negligible.

From equation (10) we easily compute $k$ for $m=6$, 8, 10, etc., obtaining $k=-20/924$; $k=-70/12870$; $k=-252/184756$; etc., respectively. But just as in the case of $m=4$, we gain considerably in simplicity and lose little in accuracy by replacing the above fractions by the simpler ones $k=-1/46$, $k=-1/184$, $k=-1/733$, etc., respectively. With these values of $k$ we obtain smoothing formulas for $m=6$, $m=8$, $m=10$, etc., which may be written as follows (for simplicity we suppress the $y$'s, writing only the coefficients):

$$(1/46)\,[1 - 6 + 15 + 26 + 15 - 6 + 1], \tag{12}$$

$$(1/184)\,[-1 + 8 - 28 + 56 + 114 + 56 - 28 + 8 - 1], \tag{13}$$

$$(1/733)\,[1 - 10 + 45 - 120 + 210 + 481 + 210 - 120 + 45 - 10 + 1], \text{ etc.} \tag{14}$$

For the values of $k$ which we have used formula (11) may still be employed with all the accuracy needed, and we find corresponding to (12), (13), and (14) respectively

$$\sum(e')^2 = .567\sum e^2; \quad \sum(e')^2 = .619\sum e^2; \quad \sum(e')^2 = .656\sum e^2.$$

# CARTESIAN EQUATIONS OF CIRCLES CONNECTED WITH A PLANE TRIANGLE

By PERRY A. CARIS, University of Pennsylvania

**1. Introduc** t is the purpose of this paper to show how to obtain, by methods b to be new, the Cartesian equations of the circum-circle, the nine-point circle, the in-circle and the ex-circles of a plane triangle. Use will be made only of the coefficients and the constant terms in the equations of the three lines upon which the sides of the triangle lie. The following notation will be adopted:

$$l_1: a_1x + b_1y + c_1 = 0$$

$$l_2: a_2x + b_2y + c_2 = 0 \qquad a_ib_k \neq a_kb_i.$$

$$l_3: a_3x + b_3y + c_3 = 0$$

shall denote the three lines. $D$ shall mean the determinant $(a_1b_2c_3)$. The cofactors of $a_i$, $b_i$, $c_i$ in $D$ $(i=1, 2, 3)$ shall be denoted by $A_i$, $B_i$, $C_i$. Also $d_t=+\sqrt{(a_t^2+b_t^2)}$ for $t=1, 2, 3$.

**2. To find the equation of the circum-circle.** Consider the equation

$$\lambda_1l_2l_3 + \lambda_2l_1l_3 + \lambda_3l_1l_2 = 0. \tag{1}$$

Since it is of the second degree in $x$ and $y$, it is the equation of some conic and this conic passes through the vertices of the triangle, for if $l_i$ and $l_k$ are simultaneously satisfied so is (1). Now (1) will be a circle if the $\lambda$'s are so chosen that the coefficients of $x^2$ and $y^2$ are equal and the coefficient of the $xy$-term vanishes, that is when

$$\lambda_1(a_2b_3 + a_3b_2) + \lambda_2(a_1b_3 + a_3b_1) + \lambda_3(a_1b_2 + a_2b_1) = 0$$

and
$$\lambda_1(a_2a_3 - b_2b_3) + \lambda_2(a_1a_3 - b_1b_3) + \lambda_3(a_1a_2 - b_1b_2) = 0,$$

whence
$$\lambda_1:\lambda_2:\lambda_3 = (a_2b_3 - a_3b_2)(a_1^2 + b_1^2):(a_3b_1 - a_1b_3)(a_2^2 + b_2^2):$$
$$(a_1b_2 - a_2b_1)(a_3^2 + b_3^2) = C_1d_1^2:C_2d_2^2:C_3d_3^2,$$

and the required equation can be obtained by substituting in (1).

**3. To find the equation of the nine-point circle.** The nine-point circle is the circum-circle of the triangle formed by the mid-points of the sides of the given triangle. Now the point of intersection of $l_2$ and $l_3$ is easily seen to be $(A_1/C_1, B_1/C_1)$ and that of $l_1$ and $l_3$ is $(A_2/C_2, B_2/C_2)$. Hence the mid-point lying on $l_3$ is

$$[(A_2C_1 + A_1C_2)/(2C_1C_2), (B_2C_1 + B_1C_2)/(2C_1C_2)].$$

The line $l_1'$ joining this point to the mid-point lying on $l_2$ is parallel to $l_1$ and therefore of the form $a_1x+b_1y+k=0$. Hence

$$-k = (a_1A_2C_1 + a_1A_1C_2 + b_1B_2C_1 + b_1B_1C_2)/(2C_1C_2)$$
$$= [C_2(a_1A_1 + b_1B_1 + c_1C_1) - C_1(c_1C_2 - a_1A_2 - b_1B_2)]/(2C_1C_2).$$

The value of the first parenthesis is $D$. Also $a_1A_2+b_1B_2+c_1C_2=0$, since the point $(A_2/C_2, B_2/C_2)$ is on $l_1$. Hence $-a_1A_2-b_1B_2=c_1C_2$. That is, $-k=(C_2D-2c_1C_1C_2)/(2C_1C_2)$. Whence $k=c_1-[D/(2C_1)]$ and $l_1'\equiv l_1-[D/(2C_1)]=0$. By symmetry, $l_2'\equiv l_2-[D/(2C_2)]=0$, $l_3'\equiv l_3-[D/(2C_3)]=0$.

Now the $\lambda$'s of the preceding problem do not depend on the constant terms of $l_1$, $l_2$, $l_3$. Also $l_i'$ differs from $l_i$ only in the constant term. Hence the $\lambda$'s for the nine-point circle are proportional to the same set of numbers as those of the circum-circle. That is, the equation of the nine-point circle can be obtained by replacing every $l$ in (1) by the corresponding $l'$.

**4. To find the equations of the in-circle and the ex-circles.** Consider the equation

$$\lambda_1^2l_1^2 + \lambda_2^2l_2^2 + \lambda_3^2l_3^2 - 2\lambda_1\lambda_2l_1l_2 - 2\lambda_2\lambda_3l_2l_3 - 2\lambda_1\lambda_3l_1l_3 = 0. \tag{2}$$

It is plainly a conic touching $l_i$ for $l_i=0$ reduces it to a perfect square. This conic will be a circle if

$$\lambda_1^2a_1b_1 - \lambda_1\lambda_2(a_1b_2 + a_2b_1) + \lambda_2^2a_2b_2 - \lambda_1\lambda_3(a_1b_3 + a_3b_1) - \lambda_2\lambda_3(a_2b_3 + a_3b_2) + \lambda_3^2a_3b_3 = 0 \tag{3}$$

and

$$\lambda_1^2(a_1^2 - b_1^2) + \lambda_1\lambda_2(2b_1b_2 - 2a_1a_2) + \lambda_2^2(a_2^2 - b_2^2) + \lambda_1\lambda_3(2b_1b_3 - 2a_1a_3) + \lambda_2\lambda_3(2b_2b_3 - 2a_2a_3) + \lambda_3^2(a_3^2 - b_3^2) = 0. \tag{4}$$

Multiplying (3) by $2b_1b_2-2a_1a_2$ and (4) by $a_1b_2+a_2b_1$ and adding gives

$$C_3d_1^2\lambda_1^2 - C_3d_2^2\lambda_2^2 + 2C_1d_1^2\lambda_1\lambda_3 - 2C_2d_2^2\lambda_2\lambda_3 + [(C_1^2d_1^2 - C_2^2d_2^2)\lambda_3^2/C_3] = 0,$$

which when multiplied by $C_3$ can be broken up into two linear factors, giving

$$C_3d_1\lambda_1 + C_3d_2\lambda_2 + (C_1d_1 + C_2d_2)\lambda_3 = 0 \tag{5}$$

$$C_3d_1\lambda_1 - C_3d_2\lambda_2 + (C_1d_1 - C_2d_2)\lambda_3 = 0. \tag{6}$$

To eliminate the $\lambda_1\lambda_3$-term between (3) and (4), interchange the subscripts 2 and 3, obtaining:

$$C_2d_1\lambda_1 + (C_1d_1 + C_3d_3)\lambda_2 + C_2d_3\lambda_3 = 0 \tag{7}$$

$$C_2d_1\lambda_1 + (C_1d_1 - C_3d_3)\lambda_2 - C_2d_3\lambda_3 = 0. \tag{8}$$

From (5) and (7),

$$\lambda_1:\lambda_2:\lambda_3 = -C_1^2d_1^2 - C_1C_2d_1d_2 - C_1C_3d_1d_3:$$
$$C_1C_2d_1^2 + C_2^2d_1d_2 - C_2C_3d_1d_3:C_1C_3d_1^2 - C_2C_3d_1d_2 + C_3^2d_1d_3$$

or

$$\lambda_1:\lambda_2:\lambda_3 = C_1(-C_1d_1 - C_2d_2 - C_3d_3):$$
$$C_2(C_1d_1 + C_2d_2 - C_3d_3):C_3(C_1d_1 - C_2d_2 + C_3d_3).$$

Now for the lines $l_1$, $l_2$, $l_3$, the following identity holds:

$$C_i^2d_i^2 + C_k^2d_k^2 - C_j^2d_j^2 = -2C_iC_k(a_ia_k + b_ib_k),$$

where $i, j, k$, are all different. The verification of the identity presents no difficulty at all. Multiplying each term of the above ratios for the $\lambda$'s by $-C_1d_1+C_2d_2+C_3d_3$ and using the above identity gives:

$$\begin{aligned}\lambda_1:\lambda_2:\lambda_3 &= C_1(C_1^2d_1^2 - C_2^2d_2^2 - C_3^2d_3^2 - 2C_2C_3d_2d_3):\\ &\quad C_2(-C_1^2d_1^2 + C_2^2d_2^2 - C_3^2d_3^2 + 2C_1C_3d_1d_3):\\ &\quad C_3(-C_1^2d_1^2 - C_2^2d_2^2 + C_3^2d_3^2 + 2C_1C_2d_1d_2)\\ &= C_1[2C_2C_3(a_2a_3 + b_2b_3) - 2C_2C_3d_2d_3]:\\ &\quad C_2[2C_1C_3(a_1a_3 + b_1b_3) + 2C_1C_3d_1d_3]:\\ &\quad C_3[2C_1C_2(a_1a_2 + b_1b_2) + 2C_1C_2d_1d_2]\\ &= (a_2a_3 + b_2b_3 - d_2d_3):(a_1a_3 + b_1b_3 + d_1d_3):(a_1a_2 + b_1b_2 + d_1d_2). \qquad (9)\end{aligned}$$

Equation (7) differs from (8) only in the sign of $d_3$. Hence the solution of (5) and (8) is

$$\lambda_1:\lambda_2:\lambda_3 = (a_2a_3 + b_2b_3 + d_2d_3):(a_1a_3 + b_1b_3 - d_1d_3):(a_1a_2 + b_1b_2 + d_1d_2). \qquad (10)$$

Also (6) differs from (5) only in the sign of $d_2$. Hence the solution of (6) and (7) is

$$\lambda_1:\lambda_2:\lambda_3 = (a_2a_3 + b_2b_3 + d_2d_3):(a_1a_3 + b_1b_3 + d_1d_3):(a_1a_2 + b_1b_2 - d_1d_2). \qquad (11)$$

Now if (5) and (7) be each multiplied by $-1$, then (6) will differ from (5) only in the sign of $d_1$ and (8) will differ from (7) in the same way. Hence the solution of (6) and (8) can be obtained from that of (5) and (7) by changing the sign of $d_1$. Hence, in this case

$$\lambda_1:\lambda_2:\lambda_3 = (a_2a_3 + b_2b_3 - d_2d_3):(a_1a_3 + b_1b_3 - d_1d_3):(a_1a_2 + b_1b_2 - d_1d_2). \qquad (12)$$

The four sets, (9), (10), (11), (12), give the ratios of the $\lambda$'s for the required four circles. It remains to show which set is to be chosen to find the equation of a prescribed circle. For this purpose, write (2) in the form

$$(\lambda_1l_1 + \lambda_2l_2 - \lambda_3l_3)^2 = 4\lambda_1\lambda_2l_1l_2. \qquad (13)$$

It may be remarked in passing that this form is to be preferred to (2) in a numerical problem. Since (13) is the equation of a conic in terms of two tangents and their chord of contact, the point of contact of $l_1$, say, is at the intersection of $l_1=0$ and $\lambda_2 l_2-\lambda_3 l_3=0$, that is at the intersection of $a_1x+b_1y+c_1=0$ and $(\lambda_2 a_2-\lambda_3 a_3)x+(\lambda_2 b_2-\lambda_3 b_3)y+\lambda_2 c_2-\lambda_3 c_3=0$, which is the point

$$\left(\frac{\lambda_2 A_3+\lambda_3 A_2}{\lambda_2 C_3+\lambda_3 C_2}, \frac{\lambda_2 B_3+\lambda_3 B_2}{\lambda_2 C_3+\lambda_3 C_2}\right). \tag{14}$$

In the problem on the nine-point circle, one vertex lying on $l_1$ was found to be $(A_2/C_2, B_2/C_2)$. The other one lying on $l_1$ is found by symmetry to be the point $(A_3/C_3, B_3/C_3)$. Now the circle (13) will touch $l_1$ internally, that is at a point *between* $(A_2/C_2, B_2/C_2)$ and $(A_3/C_3, B_3/C_3)$, if the point (14) divides the segment between these vertices into segments whose ratio is positive,

that is, when $$\frac{\dfrac{\lambda_2 A_3+\lambda_3 A_2}{\lambda_2 C_3+\lambda_3 C_2}-\dfrac{A_3}{C_3}}{\dfrac{A_2}{C_2}-\dfrac{\lambda_2 A_3+\lambda_3 A_2}{\lambda_3 C_2+\lambda_3 C_2}}=\frac{\lambda_3 C_2}{\lambda_2 C_3} \text{ is positive.}$$

Similarly, $l_2$ will be touched internally when $(\lambda_1 C_3)/(\lambda_3 C_1)$ is positive and $l_3$ when $(\lambda_2 C_1)/(\lambda_1 C_2)$ is positive. Since the product of the three ratios $(\lambda_3 C_2)/(\lambda_2 C_3)$, $(\lambda_1 C_3)/(\lambda_3 C_1)$, $(\lambda_2 C_1)/(\lambda_1 C_2)$, is unity, there are four and just four possibilities of signs:

(15) $+\ +\ +$ (16) $+\ -\ -$

(17) $-\ +\ -$ (18) $-\ -\ +$

Hence that one of the four sets of $\lambda$'s which produces (15) when combined with the $C$'s as shown above will give the equation of the in-circle. The sets similarly producing (16), (17), (18), will give the equations of the ex-circles touching internally $l_1$, $l_2$, $l_3$ respectively. That the $\lambda$'s and $C$'s always will produce exactly the above four sets of signs can be seen from the following consideration.

It is an easy matter to show that the absolute value of $a_i a_k+b_i b_k$ is less than $d_i d_k$. Hence, using actual values in (9), (10), (11), (12), instead of ratios, $\lambda_j$ has the same sign as the $d_i d_k$ associated with it. It is then merely a question of testing the sets of $\lambda$'s with the eight possibilities of sign that the $C$'s present.

# THE DUST NUMERALS AMONG THE ANCIENT ARABS

By DAVID EUGENE SMITH, Teachers College, Columbia University
and SALIH MOURAD, Robert College, Constantinople

Among the books composed by Eastern Moslem scholars in the tenth century, several have special interest with respect to elementary computation and the gobar (ghobar) numerals. One was written by 'Alî ibn Aḥmed, Abû'l-Qâsim, al-Antâki (that is, from Antioch), al-Mujtabâ (the chosen), who lived at Bagdad and died in 987.[1] It was called *Kitâb al-takht al-kabîr fî'l ḥisab al-Hindi* ("The great book of the board on Hindu arithmetic"). The word *takht* was translated at first, in Suter's list, as "the method" and later as "the table,"[2] but it is merely an Arabic form of the Persian word *takhta*, meaning "board" or "wood."[3] The same word appears in other treatises[4] by the same writer, and in works by various other writers. One of these is Mohammed ibn 'Abdallâh, Abû Nasr, al-Kalwâdânî, a contemporary of al-Mujtabâ and, like him, living at Bagdad. His work is entitled *Kitâb-al-takht fî'l ḥisâb al-Hindi*, meaning "the book of the board (takht) or the Hindu arithmetic."[5]

Another writer using the term is Sinân ibn al-Fatḥ (c. 941), a native of Harrân and a mathematician of some merit. The title of this work is *Ilm ḥisâb al-takht* ("The science of arithmetic of the takht").[6] Besides this there may be mentioned the *Kitâb al-takht* ("Book of the board") by al-Razi (ar-Razi) whose date is unknown; the *Kâfiya fî ḥisâb al-takht w'al mîl* ("Compendium of arithmetic with board and stylus"), by Amîn al-Dîn (eddîn) al-Abharî, who died in 1332[7]; and the *Jawâmi' al-ḥisâb bi'l takht w'al-turâb* ("Encyclopedia of arithmetic with board and dust"), written by 'Abdallah al-Zanâtî whose date, like that of al-Razi, is unknown.

The derivation and significance of the word *takht* seem more reasonable than those advanced by Woepcke, Suter, and others of their time. Just as we

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[1] *Fihrist*, p. 40, and p. 75, nn. 271–276; Arabic text, vol. 1, p. 284; Suter's list, p. 63. See Casiri, *Bibliotheca Arabico-Hispana*, vol. 1, p. 411; Weissenborn, *Zur Geschichte der Einführung der jetzigen Ziffern in Europa*, p. 87. The transliterations and translations in the text and the notes have been checked and the references in the notes extended by Dr. Soloman Gandz, whose writings on Arabic mathematics are familiar to readers of this Monthly.

[2] This was due to the fact that he misread the Arabic letter *kh* for *ḥ*. See p. 70 and his note 232 on the *Fihrist*.

[3] See Vullers, *Lexicon Persico-Latinum*, vol. 1, p. 425; Lane, p. 298.

[4] *Kitâb fi'l ḥisâb 'alâ al-takht bilâ maḥw* ("The book on arithmetic on the board without erasing"). The *Fihrist*, *loc. cit.* Arabic title *Kitâb al-ḥisâb bilâ takht bal bi'l-yad* ("The book on arithmetic with the hand without the board"). This refers to mental arithmetic. See *Fihrist*, p. 70, on the authority of Ibn al-Ḳifṭi (1172–1248): See his *Ta'rikh al-Ḥukamâ*, ed. Lippert, Leipzig 1903, p. 234.

[5] *Fihrist*, pp. 41, 75; Arabic text, vol. 1, p. 284; Suter's list, p. 74, and Ibn al-Ḳiftî Ta'rikh al-Hukamâ p. 288.

[6] The *Fihrist*, pp. 37, 70; Arabic text, p. 281; Suter's list, pp. 51 and 66.

[7] See Ahlwandt's catalogue of Arabic MSS in Berlin, vol. 5, p. 333, No. 5975.

send a pupil to "the board," meaning the blackboard, so the Arabs of the 10th century sent them to their board tablets, as may also be seen in most of the native Arab schools today. In that century, however, when writing materials, as we know them, were scarce, the pupils sprinkled white dust on a black tablet and did their computing with a stylus or pointed rod. Hence it is that later Arab writers speak of "arithmetic on the dust board," of "arithmetic on the board with the pointer," of "*al-ghobar* (dust) arithmetic," and (as in the case of Sinân ibn al-Fatḥ, of "the science of the board."

The use of the word *takht* is more fully explained in the bibliographical lexicon of Haji Khalfa[1] as follows:

"There are many divisions of arithmetic. One of them is the science of arithmetic of the board and the stylus,[2] *al-takht w'al-mîl*. This is the science by which we learn how to perform the arithmetic operations by numerical symbols for the units and also for numbers exceeding the units by the stages.[3] These symbols are attributed to the Hindus. . . . . This science as also called, *al-takht w'al-turâb* ("The board and the dust")."

The form *takht* should be used instead of *taḫt*, although the *Fihrist* and Haji Khalfa adopted the latter. The *ḫ* is the Arabic letter *h* without a dot, whereas *kh* represents the same letter with a dot and is more gutteral than the other. *Takht* is therefore the more correct form and is properly used by both Ahlwardt and Woepcke.

The eastern Arabs kept the name of Hindu[4] arithmetic, while those in western Europe preferred the title *al-gobar*, or *al-ghobâr* (dust) as characterizing their computations.[5] It is not strange that the Arabs in Spain should have continued to use the dust tablet after it was abandoned in the East, for paper was made in Bagdad as early as the ninth century, but the industry was not carried to Spain until three or four centuries later.[6] It is also quite natural that they should have used the word to mean abacus in general, long after the dust feature was forgotten.

There seems, therefore, to be no doubt that, although the western Arabs appropriated the term *gobar* (*ghobar*), the early scholars of Bagdad used the

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[1] *Lexicon Bibliographicum et Encyclopaediam*, ed. of the Arabic text with Latin translation by G. Fluegel, in 7 vols., Leipzig, 1835–1858.

[2] See Smith, *History*, vol. 2, pp. 177–178, "radius."

[3] Or "by the degrees," referring to place value.

[4] But see Professor Carra de Vaux's argument that *al-Hindi* did not necessarily mean Hindu.

[5] Dr. Gandz, whose paper on this subject will be published later, takes *ghobar* to be simply an Arabic transliteration of the Greek term *abakion* (abacus),—the Semitic *abaq*; but this simply carries it back to a word meaning dust, as before.

[6] It dates from the beginning of the Christian era in China; at any rate our first specimens of paper go back to the first century. The industry was introduced into Samarkand by Chinese prisoners of war in 751, and a paper factory was established in Bagdad in 794. The first one set up in Spain was in 1154.

dust numerals in the same way as the later computers of Cordova, Salamanca, and Toledo.[1]

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## QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

**The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.**

### DISCUSSIONS

### I. CONVERGENCE WITHOUT LIMITS

BY A. A. BENNETT, Lehigh University

The following question is sometimes proposed to catch the unwary mathematician. Suppose $f(x)$ and $g(y)$ are two real one-valued bounded functions defined for all real values of their arguments, $f(x)$ being furthermore continuous. Suppose $a$, $b$, $c$, are numbers satisfying the conditions, $\lim_{x\to a} f(x)=b$, $\lim_{y\to b} g(y)=c$; the question is, does it follow that $\lim_{x\to a} g[f(x)]=c$? The correct answer is of course "No," since the definition of $\lim_{y\to b} g(y)=c$, does not make use of $y=b$, and on the other hand the definition of $\lim_{x\to a} f(x)=b$, does not require that $f(x)\neq b$. Thus, for example, if $f(x)$ is taken as a constant, namely $b$, and $g(y)$ is discontinuous at $y=b$, but has a limit as $y$ approaches $b$, then the proposed relation cannot hold. For example let $f(x)\equiv b$, and let $g(y)\equiv c[\sin(y-b)]/(y-b)$, for $y\neq b$, but let $g(b)=c-l$. Then $\lim_{y\to b} g(y)=c$, but $g[f(x)]\equiv c-l$, so that $\lim_{x\to a} g[f(x)]\neq c$.

In view of these facts it may not seem so much of a paradox to assert that the statement: "For a given $a$, a value $b$ exists such that $\lim_{x\to a} f(x)=b$," may be altered without for many purposes essentially impairing its content and yet in such a way as not to assume the existence of either $a$ or $b$. One might naturally ask as to what if anything would be gained by such a procedure even if it is possible. Another question somewhat more concrete is the following: Is there anything to be gained by sacrificing the use of the continuity of the real number system?

One might raise the question concerning the possibility of making actual use of more than an enumerable set of numbers. So far as any possible explicit

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[1] After this article was prepared, it was submitted to Dr. Soloman Gandz, as stated in a previous note, who was thereby led to make an extended study of the question of the abacus in general among the Arabs. His investigations supplement in various instances the treatment of the special topic as here presented, but they carry the subject so much farther that he has been asked to publish the results of his study in a subsequent number of this MONTHLY.

numerical relation is concerned it is clear that the Dedekind-cut axiom on numbers places at our disposal a nondenumerable set of irrational numbers "almost all" of which are never used. However more elementary considerations may be cited which would suggest the desirability in some cases of omitting some limiting points. Consider the theory of the set of points in the Euclidean plane, constructible from two given distinct points by use of ruler and compasses, only points obtained as intersections being counted. After constructing the vertices of a regular hexagon with one of the points as a center by the use of compasses, we can, by employing the ruler alone, secure a complete rational planar net. The further use of compasses brings in further points. For the complete system of points, quadratic equations with positive discriminants have solutions but cubic equations for example are not in general solvable, and a Dedekind cut among the points of a line may not determine a point of the system. For the totality of constructible points, convergence has a meaning, but limits do not in general exist. One might desire to introduce the limiting points. The distinction between constructible and non-constructible points is not thereby abolished. And the limiting points would continue to have a minor role at best in the theory of geometrical constructions.

In certain other illustrations, the limiting points would be worse than useless. Consider for example the following. Let there be given a rational planar net of points. Let Cartesian coordinates be introduced and suppose that each point of the net is designated by an ordered pair of rational numbers $(a, b)$. In the general theory of such planar nets, a notion of normal region and hence of convergence may be introduced as for example by Minkowski in his geometry of numbers, or much more abstractly by Hausdorff and Fréchet in their discussions of general spaces. Any such theory would be developed as applicable to all rational planar nets using the given form of region. Suppose we should particularize and consider a geometrical representation of the numbers in the algebraic field generated by the square root of 2. Let each number of the form $a+b\sqrt{2}$, where $a$ and $b$ are rational, be represented by the point $(a, b)$ in the net. Convergence may be discussed with reference to a notion of absolute value when we define arbitrarily Abs $(a, b)=\sqrt{(a^2+b^2)}$, and we may denote this by $||(a, b)||$. This will satisfy the triangle inequality needed for convergence, although for most algebraic purposes it will not be as useful as the familiar norm as used in the theory of algebraic fields, which latter fails to comply with this condition of inequality. Consider the sequence (0,1), (0,1.4), (0,1.41) etc., where $(0,b_n)$ is such that $\lim_{n\to\infty} b_n=\sqrt{2}$. If a limiting point for this sequence exists, one would expect it to be a point $(0,\sqrt{2})$, but this would denote $0+\sqrt{2}\sqrt{2}=2=(2,0)$. Thus convergence and limit relations are seriously disordered by supposing any such a limit as existing.

The general question as to how one can handle convergence and relative continuity in any domain where compactness is sacrificed presents no great logical difficulty but offers several features of immediate interest. Let us define a sequence $(x_n)$ as convergent if for each positive real number, $\delta$, there is a natural number $m$, such that $||x_n - x_{n+p}|| < \delta$ for each $n \geqq m$ and each $p > 0$. We shall define such a number $m$ as *belonging* to $\delta$ for the given sequence. A function $f(x)$ will be defined as convergent with respect to a convergent sequence $(x_n)$ if and only if for each positive real number, $\epsilon$, there is a positive real number $\delta$, such that for an $m$ belonging to $\delta$ as defined by the reference sequence, $||f(x_m) - f(x)|| < \epsilon$, for each $x$ in the domain for which $||x_m - x|| < \delta$. One may note the close resemblance between this definition and the usual condition that $f(x)$ has a limit for $x$ approaching a limit, although here no limits are required. In particular it follows at once that the sequence $(f(x_n))$ converges. However, it is to be noted that the definition is not for a sequence of values of the function. Using this definition one may show that among rational numbers $f(x) \equiv x^3$ converges as $x$ converges toward the cut usually defined as $\sqrt{2}$, although neither limit exists in the domain.

In the usual discussions of general spaces, the notion of compactness which makes possible the introduction in one dimension of a Dedekind cut, and of a Borel theorem is usually regarded as so important that the continued study of convergent sequences has seldom if ever been carried to the extent of considering the notion of the convergence of functions of a variable point of the discontinuous space. Thus for example both Pierpont and Pringsheim who carry out the Cantor theory to an elaborate degree introduce the complete real number system before venturing upon functions of an independent variable. While for many purposes this is undoubtedly wise, yet for certain algebraic problems the more general treatment here proposed has distinct advantages.

## II. The Devil's Curve Again

By Aubrey J. Kempner, University of Colorado

The text of P. R. Rider's article "The Devil's curve and Abelian Integrals"[1] and of B. H. Brown's note "La courbe du diable"[2] may give the impression that the curve is not easy to trace. It may be worth while to point out that the most elementary rules for curve tracing bring this curve within the scope of freshman analytic geometry, without any use of calculus. The special case $y^4 - x^4 - y^2 + 2x^2 = 0$ (considered in the first article mentioned above) would be plotted by the following steps:

$$Y^2 - X^2 - Y + 2X = 0, \quad \frac{(X-1)^2}{3/4} - \frac{(Y-\frac{1}{2})^2}{3/4} = 1.$$

[1] This Monthly, vol. 34 (1927) pp. 199.

[2] This Monthly, vol. 33 (1926) pp. 273–274.

Plot this hyperbola in an $X$-$Y$ system of coordinates. Then take the square root of the $X$-coordinate ($x^2=X$), and in this new figure (in an $x$-$Y$ system of coordinates) take the square root of the $Y$-coordinate ($y^2=Y$). Fig. 1 of Rider's paper is thus immediately obtained. (Compare, for example, this Monthly, vol. 24 (1917), p. 18).

It goes without saying that the curve actually does all that is claimed for it as far as the illustration of many properties (singular points, maxima and minima, asymptotes, etc.) is concerned, but it might have been mentioned how easy it is to plot the curve itself.

## III. An Example in Maxima and Minima

By Elijah Swift, University of Vermont

It sometimes puzzles the beginning student to find functions of two variables, $x$ and $y$, which possess a minimum in $x$ alone and also in $y$ alone without possessing one in both variables. Thus, if we set $x=0$, the expression $x^2-3xy+y^2$ (which then becomes a function of $y$ alone, namely $y^2$) takes on a smaller value, namely 0, for $y=0$ than for any other value of $y$ and the same thing is true for $x$ when we set $y=0$; but the original expression may take on negative values for pairs of values of $x$ and $y$ arbitrarily near the origin, e. g. for $x=\epsilon$, $y=\epsilon$, and consequently does not possess a minimum at (0,0).

In this connection it may be interesting to exhibit a function of $n$ variables which has the property that it does not possess a minimum in all these, but is such that if we set any one of them equal to zero it will possess a minimum in the remaining $n-1$. Such a function is

$$(2x_1 + x_2)^2 + (2x_1 + x_3)^2 + \cdots + (2x_1 + x_n)^2 - x_1^2 .$$

A moment's inspection shows that if we set any variable, say $x_n$, equal to zero, the above expression will have a positive value for all values of the remaining variables not all zero, while it vanishes if all the variables vanish. On the other hand if we set $x_1=\epsilon$, $x_2=x_3=\cdots=x_n=-2\epsilon$, the function is negative and consequently does not have a minimum for the values $x_1=x_2\cdots=x_n=0$.

## IV. A Note on the Sources of Mathematical Reality

By Alan D. Campbell, University of Arkansas

In J. B. Shaw's *Lectures on the Philosophy of Mathematics*, page 10, we find the statement that the sources of mathematical reality have been ascribed at various times to four different worlds:

1. The world of natural phenomena.
2. The world of universals.
3. The world of mental activity.
4. The creative action of the intellect.

Shaw points out (pages 39, 40) that the fourth and higher dimensions and the imaginary numbers are some of the entities that preclude mathematics from being derived entirely from the world of natural phenomena, although mathematics has some of its roots in nature. Also he notes (pages 44, 192) that our inability to choose definitely between Euclidean and non-Euclidean geometries to describe our physical space (especially now with the advent of relativity) and the arbitrariness of mathematical constructions are two of the aspects of mathematics that prove it is not bound up entirely with the world of mental activity (not one of Kant's "categories" of the mind). Here we have to admit that mathematics (and all reasoning) is dependent in part on the structure of the human mind. Thus we can conceive of the Martians as having different sorts of minds from ours, and so a different logic and a different mathematics.

The purpose of this note is to call attention to the apparent balance between the arguments in favor of the world of universals as the main source of mathematical reality and those in favor of the creative action of the intellect as the main source. For example one may call attention to the fact that whenever a theorem in mathematics is lost and rediscovered (or is discovered by several mathematicians independently), this theorem has always practically the same form. This seems to point toward mathematics as existing in a Platonic world of universals where this theorem was lost and found again. But this similarity of form may on the other hand be due to the uniform structure of the human mind. Again a mathematician feels, when he fashions a new method to solve a problem he has undertaken, that he has created this weapon of the intellect. But this feeling that he has invented something new may be deceptive, just as the feeling that we are absolutely free agents in our choices of actions is deceptive (because psychology shows us that habit and bodily states affect such choices). So we may read Shaw's book and Keyser's book on mathematical philosophy and any other books or papers there are that bear on this subject, and we shall find that the arguments for and against each of these two worlds as the main source of mathematical reality balance one another, and that any objection that can be raised against one or the other of these two sources can be answered (as the above objections were answered).

The point we wish to make now is that in all probability there never will be found any crucial tests to decide between the two conflicting sources, but that the whole question will remain one of individual choice (due to individual psychological differences). Thus one mathematician may be more inspired in his work by the feeling that he is an explorer in a "world of ideals," another may be more thrilled by the idea that he is an inventor (or creator) of mental weapons for the conquest of nature. In fact one and the same person may hold first to one and then to the other view of the matter at different times. He may

use the "creation of the intellect" theory as a working hypothesis, but when he comes to ponder on the question may decide in favor of the other source; then again he may later be drawn back to bow in reverence before the majesty of the human mind that can fabricate such a marvel as mathematics.

## V. A Theorem on Improper Definite Integrals

By H. L. Slobin, University of New Hampshire

Theorem: If $f(x)$ is a function integrable from $-\infty$ to $\infty$ and if $f(x-x^{-1})$ is also integrable from $-\infty$ to $\infty$, then

$$\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Proof: Let $f(x)$ be an even function $\phi(x)$ so that $\phi(x)=\phi(-x)$. Let $z=1/x$. Then

$$\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{z=\infty}^{0} \phi\left(\frac{1}{z}-z\right)d\left(\frac{1}{z}\right)$$

$$= \int_{z=\infty}^{z=0} \phi\left(\frac{1}{z}-z\right)d\left(\frac{1}{z}-z\right) + \int_{z=\infty}^{z=0} \phi\left(\frac{1}{z}-z\right)dz.$$

Let $(1/z)-z=u$. Then $u=-\infty$ where $z=\infty$, and $u=+\infty$ when $z=0$. Hence

$$\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi(u)du - \int_0^{\infty} \phi\left(\frac{1}{z}-z\right)dz$$

or

$$2\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi(u)du. \tag{1}$$

But

$$2\int_0^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi\left(x-\frac{1}{x}\right)dx$$

because $\phi(x)$ is an even function. Therefore (1) becomes

$$\int_{-\infty}^{\infty} \phi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \phi(x)dx. \tag{2}$$

This completes the proof when $f(x)$ is an even function. Now let $f(x)$ be an odd function $\psi(x)$ so that $\psi(x)=-\psi(-x)$. Then

$$\int_{-\infty}^{\infty} \psi\left(x-\frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \psi(x)dx = 0, \tag{3}$$

because the integral of any odd function from $-\infty$ to $\infty$ is equal to zero.

It remains to prove the theorem for the function $f(x)$ of the theorem. That function can be expressed as the sum of an even function $\phi(x)$ and an odd function $\psi(x)$ as follows:

$$(4) \qquad f(x) = \phi(x) + \psi(x) \text{ where } \phi(x) = \tfrac{1}{2}[f(x) + f(-x)] \text{ and } \psi(x) = \tfrac{1}{2}[f(x) - f(-x)].$$

Then

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right)dx = \int_{-\infty}^{\infty} \left[\phi\left(x - \frac{1}{x}\right) + \psi\left(x - \frac{1}{x}\right)\right]dx, \text{ by (4)}$$

$$= \int_{-\infty}^{\infty} [\phi(x) + \psi(x)]dx = \int_{-\infty}^{\infty} f(x)dx, \text{ by (2), (3), (4).}$$

This completes the proof of the theorem.

QUESTION: It is desired to know if there are any practical applications of the theorem and what, specifically, are the restrictions which it imposes on the function $f(x)$.

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## RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

### REVIEWS

### THE SECOND CARUS MONOGRAPH

*Analytic Functions of a Complex Variable.* By D. R. CURTISS. Chicago, The Open Court Publishing Company, 1926. ix+ 173 pages. Price $2.00.

This is the second volume to appear in the series of *Carus Mathematical Monographs*. The aim of the series may fairly be termed unique in the history of mathematics. While possible usefulness to mathematical and scientific students is kept in mind, the chief purpose is to make accessible to thoughtful and educated people generally some of the most important and fruitful results in the field of mathematics.

Every mathematician has probably felt at some time or another a sense of hopelessness about trying to satisfy the intellectual curiosity of others regarding his own work. There are public lectures and printed articles without number by which this curiosity may be satisfied regarding special topics of interest in almost any scientific or intellectual movement other than mathematics. But the queen of the sciences, the one that is essential for the highest development of almost every other exact science, communicates its ideas and results in a language that must be studied to be understood. In addition to this, the full significance of a mathematical theory or formula can be obtained,

in many cases, only by prolonged thought, after examining in detail all the steps of the reasoning and verifying the algebraic processes that accompany them. All of this requires technical skill to be obtained only by long training. How, then, is it possible to "popularize" any mathematical theory in even the slightest degree? It is obvious that this cannot be done except possibly for a comparatively small number of persons who have some knowledge of the elements of mathematics, i. e., of algebra, geometry, trigonometry, analytic geometry, and calculus. The task is certainly a difficult one at best, and only by the most thoughtful labor on the part of those who are willing to lend their aid can this series of monographs hope to satisfactorily fulfill its larger purpose. It is, therefore, with especial interest that mathematicians will note the efforts in this direction as they appear from time to time in published form.

No more interesting subject could have been chosen for this new volume than that indicated in the title. Its importance has been characterized by Hadamard in a very recent statement: "The main, and central, subject of modern analysis lies in the two connected theories of functions and of differential equations."

The task of putting a theory of such importance and magnitude in a nutshell, where "he who runs may read," is obviously one of great difficulty. It has been performed so well in this monograph that there seems little need for the reviewer to do more than express his sincere admiration for the completed result. Good judgment has been exercised in the delicate task of selecting material so as to give a clear notion of some of the fundamental and characteristic ideas and methods in this theory. The proofs are skillfully handled so as to reduce the algebraic work to a minimum of detail and to make the essential steps as simple, clear-cut, and effective as possible. The author sticks closely to a well-ordered plan of treating only the most vital and intimately connected principles of the theory, resisting the ever present temptation to make side excursions for inspecting the vast and exceedingly interesting and varied details presented by special classes of functions. Only such illustrative material is used, and that of the simplest kind, as is absolutely necessary to clarify fundamental concepts or general methods.

The author very properly gives the ideas, methods, and formulas of Cauchy predominance in his treatment, but the present-day view point in modern analysis pervades the entire book. The influence of Weierstrass comes to the front in the treatment of series and analytic continuation, while the fertility of Riemann's ideas is brought out with great clearness in the consideration of many-valued functions.

The second chapter, on complex numbers, is the only place where the reviewer would wish to see any change. In his opinion it could advantageously be shortened by omission of the graphical representation of multiplication

and division. It is unessential to the book and is no aid whatever to the study of these operations on complex numbers. It should be dropped from our text books and from our teaching as useless lumber. The algebraic theory of these operations by use of polar coordinates is perfectly simple and intelligible and all that is needed.

It would seem especially desirable to make the formula work in the beginning of such a monograph as simple and brief as possible. I am, therefore, inclined to think it would have been better to explain multiplication, division, and extraction of roots by using the exponential form $\epsilon^{i\theta}$ instead of $\cos\theta+i\sin\theta$ referring ahead to pages 53–4 for a proof of the equivalence of these expressions.

An admirable feature of the book consists in giving, at the end of each chapter, full and explicit page references to several standard treatises for additional reading on the various topics of the chapter.

The book is very attractively printed and bound. The typographical errors are extremely few. In the second line below the formulas at the middle of page 45, the words, "as $Z$ approaches $z$ is $f(z)$," should read "as $z$ approaches $Z$ is $f(Z)$." In formula (10), p. 127, the first two subscripts of the coefficients should be $-2$ and $-1$.

With so simple and attractive an exposition of the elements of the theory of analytic functions of a complex variable now available, it is earnestly to be hoped that many who are not yet familiar with this beautiful theory will read this little volume and derive from it some idea of one of the great conquests in the field of mathematics.

To make this task as easy and pleasant as possible, it is suggested that Chapter II be omitted entirely on first reading by those who have any knowledge of complex numbers, and that it be used only for reference as occasion may arise. For those who have too little familiarity with complex numbers, a study of the first five pages of Chapter II together with Article 7 would be sufficient for the reading of almost everything that follows in the book. Finally, we might note the author's suggestion in the preface, that, even without a knowledge of the calculus, the general reader may "obtain some idea of the scope and purposes of the theory of functions" from an examination of the book.

Nothing has yet been said about the usefulness of this book to the mathematical student or teacher. From what precedes, it may easily be inferred that no better introduction to the subject could be found. A mastery of the contents of this book ought to make further study of functions of a complex variable easy and rapid. It is believed, also, that the teacher will find it an ideal text for a very brief course, especially for a Summer School class.

J. I. HUTCHINSON

*Nicomachus of Gerasa, Introduction to Arithmetic.* Translated into English by MARTIN LUTHER D'OOGE, with Studies in Greek Arithmetic by FRANK EGLESTON ROBBINS and LOUIS CHARLES KARPINSKI. New York, The Macmillan Co., 1926. ix+318 pages.

This volume is a beautiful illustration of the advantages of cooperative effort in research. The late Professor D'Ooge performed the translation of the arithmetic of Nicomachus from the Greek, but left the manuscript unfinished. This phase of the work was continued by Mr. Robbins who also wrote the chapter on the development of Greek arithmetic before the time of Nicomachus. The other mathematical parts are by Professor Karpinski who wrote on the sources of Greek mathematics, on Greek notation, and on the content of the Arithmetic of Nicomachus.

A fitting ornament to the publication is a generalization of the theorem of Nicomachus, that cubical numbers are always equal to the sum of successive odd numbers. Here again cooperative effort displayed itself. Karpinski gave the analogous theorem for fifth and the seventh powers, and generally for the $n^{2k+1}th$ power. Then Professor N. Anning gave a further extension, and Mr. E. B. Escott rose still higher in the general view-point, by a theorem which embraces as special cases the generalizations of Karpinski and Anning.

Nicomachus was not a great mathematician, but his *Introduction to Arithmetic* was widely read; it was used by Boethius and indirectly supplied the content of arithmetical study in Europe down to the time of the influx of Hindu-Arabic arithmetic. It is a work of importance to students of the history of mathematics. The men who cooperated in the production of this translation and commentary have rendered a real service to such students.

FLORIAN CAJORI.

*Calculus*, second edition. By H. W. MARCH and H. C. WOLFF. New York, McGraw-Hill Book Company, 1926. xvii+398 pages. Price $2.50.

This text is equally adapted to technical students and to others pursuing a course in calculus for four hours a week throughout the year. The subject is presented in a simple and direct manner and the more difficult topics are reserved till the later chapters.

The anti-derivative is introduced in chapter III and simple applications are given early so that courses requiring calculus as a prerequisite may be studied at the same time. The power function is treated early with applications to curves and moving bodies. Implicit differentiation properly appears here. With algebraic functions come applications to maxima and minima and rates. Inflections and simple areas come next with work of a variable force, discussion of the parabolic cable, and further applications to mechanics.

The treatment of infinitesimals starts in chapter VII and "in this connection Duhamel's theorem is used as a valuable working principle, though the refinements of statement upon which a rigorous proof can be based have not been given" (Preface). Too much rigor in a first course tends to kill the student's enthusiasm but is there any reason why this theorem could not be stated and explained as in Osgood's *Introduction to the Calculus*, 1922, pp. 301–304? In a first course there is no valid objection to proving that the anti-derivative of a function may be represented as an area and omitting Duhamel's theorem. There is another road open to those who wish to avoid the use of the law of the mean in establishing the fundamental theorem of integral calculus. That is to use the method of proof given in Snyder and Hutchinson's *Calculus*, 1912, pp. 270–273, or in Edwards' *Integral Calculus for Beginners*, 1920, pp. 14–16.

One of the few obscure statements of the book occurs at the beginning of chapter VIII on circular functions. A reference to §55 instead of §56 was made on the same page in making the changes for the revision. The revision is a distinct improvement and includes a valuable rearrangement of chapter XVII on infinite series, additional or new exercises in chapters III to VII and IX. Certain other changes were made in the text, which now includes the new sections 61 on approximate relative error and 73 on weighted mean.

In chapter IX one is attracted by the simplicity and shortness of the method of obtaining the derivative of an exponential function. Herein lie two weaknesses: first, the existence of the limit involved is not even made plausible; secondly, no reason is given to show that the $e$ there obtained is the base of the natural system of logarithms.

Among the topics reserved till later are polar coordinates, concavity, successive integrations, special methods of integration, improper integrals, curvature, evolutes and envelopes. Infinite series are used to evaluate indeterminate forms. DeMoivre's theorem is obtained and Rolle's theorem stated, and finally the law of the mean is proved and extended to Taylor's theorem.

A chapter on solid geometry enables the uninitiated student to take up the application of triple integration without difficulty.

The last two chapters deal with total differentiation, exact differentials and differential equations. It is noted with satisfaction that the integration of an exact differential in two variables is correctly done, but the treatment of initial conditions in the solution of differential equations is not reinforced by sufficient practical exercises. Chapters XVI and XVIII also end without lists of exercises but there are about 1700 exercises given, the answers to many of which may be obtained from the publishers in an answer book. There are about 170 illustrative exercises and 145 figures.

Inaccuracies in printing occur in a few places as in line 2, p. 258; ex. 10, p. 281, upper limit for $y$; figure 107, subscript for $C$; and in ex. 5, p. 342, part of the old fashioned factorial sign is omitted. On page 369 the phrase "we seek to find a test" occurs. There are also what might be regarded as omissions, such as Newton's method and other forms of iteration, Simpson's rule, and a rapid series for $\pi$. On the whole, however, the book appeals to the reviewer as one of the most teachable books on the calculus and one of the best texts which have so far appeared in English.

C. C. CAMP.

### ARTICLES IN CURRENT PERIODICALS.

**The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.**

**Bulletin of the American Mathematical Society,** volume 32, no. 6, November-December 1926: "Recent progress with the Dirichlet problem" by O. D. Kellogg, 601–625; "On irredundant sets of postulates" by A. Church, 626–628; "Generalization of Lagrange's theorem" by L. Weisner, 629–630; "The transformation of a regular group into its conjoint" by G. A. Miller, 631–634; "A characteristic property of minimal surfaces" by J. Douglas, 635–638; "On the convergence of trigonometric approximations for a function of two variables" by E. Carlson, 639–640; "On a tensor of the second rank in function space" by D. Jackson, 641–643; "All integral solutions of $ax^2+bxy+cy^2=w_1w_2\cdots w_n$" by L. E. Dickson, 644–648; "Nuclear points in the theory of abstract sets" by W. Sierpinski, 649–653; "On the integro-differential equation of the Bôcher type in three-space" by G. E. Raynor, 654–658; "Two way continuous curves" by G. T. Whyburn, 659–663; "On some theorems of Bôcher concerning isolated singular points of harmonic functions" by O. D. Kellogg, 644–668; "The transversality relative to a surface of $\int F(x, y, z, y', z')dx=$ minimum" by J. Douglas, 699–673; "On the extension of a method of Briot and Bouquet for the reduction of singular points" by B. O. Koopman, 674–678; "A curious irreducible continuum" by W. A. Wilson, 679–681; "On a fundamental formula in the theory of class-number relations" by E. T. Bell, 682–688; "On the functional equation $f(x+y)=f(x)+f(y)$" by M. Kormes, 689–693; "On quantifiers for general propositions" by C. H. Langford, 694–704.

**Proceedings of the National Academy of Sciences, U. S. A.,** volume 31, No. 1, January 1927: "Groups generated by two operators of order three whose product is of order three" by G. A. Miller, 24–26

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## PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

**Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.**

### PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

**3258. Proposed by H. E. Arnold, Wesleyan University, Middletown, Connecticut.**

The perpendiculars dropped from the vertices of a triangle upon the lines joining the feet of the angle bisectors to the orthocenter of the triangle meet the respective sides of the triangle in three collinear points. The line joining these three points is perpendicular to the line joining the orthocenter to the center of the inscribed circle.

**3259. Proposed by Norman Anning, University of Michigan.**

Solve the cyclic set of simultaneous equations,

$$x_1 - 2x_n = x_n^2 x_1 ; \quad x_{j+1} - 2x_j = x_j^2 x_{j+1} ,$$

where $n$ is any positive integer greater than unity and $j = 1, 2, 3, \cdots, (n-1)$.

**3260. Proposed by B. C. Keeler, New York City.**

Prove that the following series is convergent.

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots + \left\{\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right\}^{-n} + \cdots$$

**3261. Proposed by J. Rosenbaum, Milford, Connecticut.**

On the sides $AB$, $BC$, $CD$, and $DA$ of a quadrilateral $ABCD$, the points $P$, $Q$, $R$, and $S$ are taken so that $AP/AB = BQ/BC = CR/CD = DS/DA$, thus forming the quadrilateral $PQRS$.

Prove that if the two quadrilaterals are similar then they are parallelograms.

**3262. Proposed by Nathan Altshiller-Court, University of Oklahoma.**

Through two given points, real or conjugate imaginary, to draw a circle so that the ends of one of its diameters shall lie on two given lines.

**3263. Proposed by Otto Dunkel, Washington University.**

Give an elementary proof, without the use of the calculus, of the theorem that the regular tetrahedron has the greatest volume of all tetrahedrons having the same surface.

## SOLUTIONS

**3193 [3181; 1926, 278] Proposed by C. K. Robbins, Purdue University.**

Solve each of the following equations:

(1) $(p - ay)^a = c(p - by)^b$;

(2) $p - ay = ce^{ay/(p-ay)}$;

(3) $[p^2 - 2apy + (a^2 + b^2)y^2]e^{\mu} = c$ where $\mu = (2a/b)\tan^{-1}[(p - ay)/by]$

where $p = dy/dx$ and $a$, $b$, $c$ are given constants.

### Solution by the Proposer.

The three equations given above correspond to the three cases which arise when one tries to solve the linear homogeneous differential equation of the second order by expressing it as one of the first order and performing one quadrature. Thus the method for solving all three is the same. Consider (3). Taking the logarithms of both sides gives

$$\log [p^2 - 2apy + (a^2 + b^2)y^2] + \frac{2a}{b}\tan^{-1}\left(\frac{p - ay}{by}\right) = \log c.$$

Differentiating gives

$$\frac{2pp' - 2a(p + yp') + (a^2 + b^2)2y}{p^2 - 2apy + (a^2 + b^2)y^2} + \frac{2a}{b}\,\frac{\dfrac{by(p' - a) - (p - ay)b}{b^2y^2}}{1 + \left(\dfrac{p - ay}{by}\right)^2} = 0,$$

where $p' = dp/dy$. Simplifying gives

$$pp' - 2ap + (a^2 + b^2)y = 0, \text{ or } \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0,$$

since

$$p = \frac{dy}{dx} \text{ and } p\frac{dp}{dy} = \frac{d^2y}{dx^2}.$$

The solution of this equation is found by standard methods to be

$$y = e^{ax}(A \cos bx + B \sin bx).$$

If we set $A = R \cos b\psi$, $B = R \sin b\psi$, where $R$ and $\psi$ are constants, and substitute in the differential equation (3), we obtain $R = b^{-1}c^{1/2}e^{-a\psi}$, so that the solution becomes

$$y = b^{-1}c^{1/2}e^{a(x-\psi)} \cos b(x - \psi).$$

A supplementary solution of (3) is

$$y = c^{1/2}(a^2 + b^2)^{-1/2}e^{\rho} \text{ where } \rho = ab^{-1}\tan^{-1}(ab^{-1}).$$

Equations (1) and (2) can be solved in similar fashion. The results are respectively,

$$y = Ae^{ax} - c^{1/a}A^{b/a}(a - b)^{b/a-1}e^{bx} (\text{if } a \neq b), \text{ and } \frac{Ae^{ax}}{a}(ax + \log A - \log c).$$

NOTE: (1) and (2) also have the solutions $y = [c(-b)^b/(-a)^a]^{1/(a-b)}$ and $y = -c/ae$ respectively.

**3195 [3183; 1926, 278] Proposed by Nathan Altshiller-Court, University of Oklahoma.**

One of two given circles is fixed, the other rolls on a fixed straight line. Find (1) the locus of the trace of the radical axis of the two circles on their line of centers; (2) the envelope of the radical axis; (3) the locus of the limiting points of the two circles.

SOLUTION BY THEODORE BENNETT, University of Illinois.

After making a suitable choice of coordinate axes, we may write the equations of the circles in the form

$$x^2 + (y - a)^2 = r^2, \quad (x - t)^2 + y^2 = R^2,$$

where $a$, $r$, $R$ are constants, and $t$ is a variable. The equations of the line of centers and of the radical axis are

$$\frac{x}{t} + \frac{y}{a} = 1, \quad 2tx - 2ay + a^2 - r^2 + R^2 - t^2 = 0.$$

Eliminating $t$ between these equations, we find that the equation of the locus required in (1) is

$$K_1 \equiv (a - y)^2(a^2 - r^2 + R^2 - 2ay) + ax^2(a - 2y) = 0.$$

Since the equation of the radical axis is quadratic in $t$, the envelope of the radical axis is easily found by equating to zero the discriminant of this quadratic; we find the locus to be

$$K_2 \equiv x^2 - 2ay + a^2 - r^2 + R^2 = 0.$$

The equation of any circle through the intersection of the two given circles may be written

$$C \equiv (x^2 + y^2 - 2tx + t^2 - R^2) + \lambda(x^2 + y^2 - 2ay + a^2 - r^2) = 0.$$

The center of $C$ is

(a) $$\xi = t/(1 + \lambda), \quad \eta = a\lambda/(1 + \lambda).$$

$C$ is a point circle if

(b) $$t^2 + a^2\lambda^2 + [R^2 - t^2 + \lambda(r^2 - a^2)](1 + \lambda) = 0.$$

Eliminating $t$ and $\lambda$ between equations (a) and (b) we find the locus required in (3) to be

$$K_3 \equiv a\eta\xi^2 - (a - \eta)[a\eta^2 - \eta(a^2 - r^2 + R^2) + aR^2] = 0.$$

The loci $K_1$, $K_2$, $K_3$ are closely related. $K_2$ is a parabola, while $K_1$ and $K_3$ are circular cubics; $K_1$ has a double point at $(0, a)$, the center of the fixed circle; $K_3$ passes through this point simply, and is non-singular. The horizontal asymptotes of $K_1$ and $K_3$ are $y=a/2$ and $\eta=0$ respectively. $K_1$ and $K_2$ cut the $y$ axis at the same point, and have the same tangent line at this point. According as this common tangent line lies below, coincides with, or lies above the line $y=a$, the singular point of $K_1$ is isolated, is a cusp, or is a double point with real tangents.

## Note by Otto Dunkel, Washington University.

The envelope may be determined geometrically. Let $C$ be the center of the fixed circle of radius $r$, $l$ the line of centers of circles of radii $R$, and $D$ the foot of the perpendicular from $C$ on $l$. Let $m=OK$ be the radical axis of the fixed circle $r$ and the circle of radius $R$ with center at $D$, and suppose that $m$ cuts $CD$ in $O$. Consider one of the variable circles with center at $Q$ on $l$. The radical axis of $D$ and $Q$ is perpendicular to $QD$ at its middle point $M$ and it cuts $m$ at $M'$. Hence the radical axis of circles $Q$ and $C$ is the perpendicular through $M'$ to $CQ$ cutting it in $H$ and cutting $CD$ in $T$. Then $H$ is a point of the first locus. Let the perpendicular to $l$ at $Q$ cut the radical axis $TM'H$ in $P$. Then $P$ is a point on the second locus. For, if we consider a circle with a center $Q'$ very near $Q$, the radical axis of $C$ and $Q$, meets that of $C$ and $Q'$ on the common chord of $Q$ and $Q'$, and this chord is perpendicular to $l$. In the limit this chord becomes the diameter of $Q$ perpendicular to $l$ along which lies $QP$ cutting $m$ in $K$. It is now easy to show from the construction that $P$ lies on a parabola whose focus is the intersection of the parallel $FM'$ to $CQ$ through $M'$ with $CD$, the axis of the parabola. For $M'$ is the middle point of $PT$; also from the similar triangles $FOM'$ and $CDQ$ we have $FO=CD/2$ since $OM'=DQ/2$. Now $PF=FT=FO+OT=FO+PK$. Hence $P$ lies on a parabola with vertex at $O$, focus at $F$, and with a directrix parallel to $m$ and at a distance from $m$ equal to $FO$.

The locus in (1) is the pedal curve of this parabola with respect to $C$. The two limit points in (3) are easily constructed and the curves may be studied from the construction.

## Also solved by A. G. Clark.

**3197 [3185; 1926, 279] Proposed by B. C. Wong, Berkeley, California.**

Prove geometrically that the tangent to the cardioid $\rho=2a(1-\cos\theta)$ and the bisector of the vectorial angle of the point of contact meet on the cissoid $\rho=2a\sin\theta\tan\theta$.

## Solution by A. G. Clark, Colorado Agricultural College.

Consider the equal circles with centers at $A$, $B$, $C$, where the circles $A$ and $B$ are fixed and tangent at $O$, while $C$ is the generating circle of the cardioid touching circle $A$ at $D$. Denote by E the other extremity of the diameter $DC$ and by $J$ the other extremity of the diameter $OB$. If the arc $DF$ is constructed equal to the arc $DO$, the locus of $F$ is the cardioid, and the tangent to the cardioid at $F$ is $EF$. Let the bisector of the vectorial angle $FOB$ cut this tangent at $P$, the circle $B$ at $G$, and the tangent to the circle $B$ at $J$ in the point $H$.

It is immediately evident from a consideration of angles that the three right triangles $OPD$, $GOJ$, and $GJH$ are similar, and since $OD=JG$, because the angle $OAD$ equals twice angle $JOG$, the first and third triangles named are equal. Therefore, $OP=GH$ thus defining the locus of $P$ as the cissoid.

## Also solved by Michael Goldberg.

**3200 [3188; 1926, 338] Proposed by B. F. Finkel, Drury College.**

Find the equation of the curve whose radius of curvature at any point of the curve is $n$ times the radius vector to the same point.

## Partial Solution by H. W. Bailey, University of Illinois.

The curve is evidently a spiral. Substitution shows that the radius of curvature, $R$, of the logarithmic spiral $\rho=\epsilon^{a\theta}$ is a constant times the radius vector at any point on the curve. Using the given condition,

$R=n\rho$, we get $n=(1+a^2)^{1/2}$ as the equation defining $a$ in terms of $n$. It follows that the required curve is $\rho=e^{\mu}$ where $\mu=(n^2-1)^{1/2}\theta$. It will be noticed at once that the curve is real only when $n\geqq 1$, and that for $n=1$ it reduces to a circle.

NOTE BY THE EDITORS This work shows that the curve $\rho=e^{a\theta}$ is a solution; it does not show that it is the only solution.

**3203[3191; 1926, 338]. Proposed by A. A. Bennett, Lehigh University.**

Prove that each positive integer less than $2\times 10^9$, has the property of containing an odd prime factor or else of being expressible as a product (with distinct factors) of the form $\Pi_i(p_i-1)$ where each $p_i$ is a prime.

SOLUTION BY MICHAEL GOLDBERG, Washington, D. C.

A number which is not an odd prime, nor divisible by an odd prime, must be a power of 2. Since Fermat's numbers $2^h+1$, where $h=2^n$, are prime for $n=0,1,2,3,4$ (but not for $n=5$), $2^n$ is then 1, 2, 4, 8, or 16. The numbers 1 to 31 can be formed by adding these numbers, not more than one of each being necessary. Hence, $\Pi_i(p_i-1)$ can become $2^{31}=\text{antilog}(31\times 0.301030)=\text{antilog}\,9.331930>2\times 10^9$.

Also solved by F. L. WILMER.

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## NOTES AND NEWS

**Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.**

The summer meeting of the Association will be held at the University of Wisconsin on Monday and Tuesday, September 5–6, in conjunction with the summer meeting and colloquium of the American Mathematical Society. The Society sessions will be on Thursday and Friday and the colloquium lectures on Tuesday to Saturday. The joint committees on arrangments are making great preparations for entertaining the visitors and there will be opportunity for those who may so desire to spend a part of August in this beautiful lake region. For information communicate with Professor Arnold Dresden, Madison, Wisconsin.

The third Carus Monograph by Professor H. L. Rietz on "Mathematical Statistics" has just been published by the Open Court Publishing Company for the Mathematical Association of America. These monographs are printed by the University of Chicago Press, and represent the acme of excellence in mechanical make-up. This series is made possible by the generosity of Mrs. Mary Hegeler Carus. They are available to members of the Association (individual and institutional) at the base cost price of one dollar and twenty-five cents each and the entire receipts from such sales go into a special fund of the Association known as the "Carus Publication Fund." Non-members can procure their monographs only through the Open Court Company at the sale price of two dollars. It is hoped that every member of the Association

will possess the entire series. Numbers four and five are in process of preparation.

There will be a new Register of the Association published in the early Autumn. Please report all changes of address for 1927–28 to the Secretary's office. And, quite as important, please try to induce all non-members in your vicinity to apply for membership before Autumn. Application blanks may be had from the Oberlin Office.

The twenty-seventh Western meeting of the American Mathematical Society was held at the University of Chicago, April 15–16, 1927. There were more than fifty papers presented aside from a two-hour symposium lecture by Professor E. W. Chittenden on "Some phases of general topology."

The next international Congress of Mathematicans will be held at Bologna during the first ten days of September, 1928. American mathematicians appreciate the consideration shown them by the Italian mathematicians in choosing a time of year convenient for those who live on this side of the Atlantic. It is hoped that many Americans will attend the Congress. Further details will be announced later.

John Simon Guggenheim Memorial Fellowships in Mathematics have been awarded as follows:

To Dr. PHILIP FRANKLIN, Assistant Professor of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts—for a study of integral equations, orthogonal functions and their relation to almost periodic functions, principally at Göttingen, Germany, and Zurich, Switzerland.

To Dr. HARRY SCHULZ VANDIVER, Associate Professor of Pure Mathematics, University of Texas, Austin, Texas—to engage in original research in connection with Fermat's last theorem, the laws of reciprocity and related topics in the theory of algebraic numbers, and to confer with specialists in the theory, chiefly in England, Switzerland and Germany.

The Massachusetts Institute of Technology has received a grant of $233,000 from the Guggenheim Fund for the Promotion of Aeronautics; the gift will provide a building and equipment.

The University of Pittsburgh has appointed a number of engineers of the Westinghouse Company as part-time lecturers at the University in electrical and mechanical engineering, physics, and engineering mathematics. Among these lecturers is Dr. JOSEPH SLEPIAN.

Professor S. D. WICKSELL of Lund University has been invited to lecture on mathematical statistics at the University of Michigan during the coming academic year.

Mr. S. F. BIBB of the University of North Dakota has been appointed professor of mathematics at the Armour Institute of Technology.

Professor ARNOLD DRESDEN of the University of Wisconsin has been appointed professor of mathematics at Swarthmore College, Swarthmore, Pa. An interesting feature of his work in that college will be in connection with the honors course for the juniors and seniors. Students in that course are not obliged to attend classes, are free to work at tasks assigned to them on which they have conferences with their instructors as often as may seem desirable. No grades or records are kept during these two years. At the end of the senior year they have to take a comprehensive examination covering the work of these two years and conducted both in its oral and written parts by an outsider.

Dr. I. MAIZLISH, Associate Professor of Mathematics and Physics and Head of the Department of Physics at Centenary College, has been promoted to a full professorship. Also he was elected president of the Louisiana Academy of Sciences on March 5th. Dr. H. L. SMITH, of Louisiana State University, is Vice-president; Prof. S. T. Sanders, of L. S. U., is a member of the Executive Council; and Prof. John A. Hardin, of Centenary College, is a member of the membership committee.

Professor E. D. MEACHEM has been made Assistant Dean of the College of Arts and Sciences at the University of Oklahoma.

Dr. A. D. MICHAL, National Research Fellow at Princeton University, has been appointed assistant professor of mathematics at the Ohio State University.

Professor EUGENIE M. MORENUS of Sweet Briar College, who has been awarded the Anna B. Brackett Memorial Fellowship by the American Association of University Women, expects to study during the coming year at Cambridge, England.

Professor J. J. NASSAU, of Case School of Applied Science, has been granted a year's leave of absence, beginning next September, to study at Cambridge University.

Dr. GEORGE E. RAYNOR, Wesleyan University, has been appointed associate professor of mathematics at the University of Oklahoma.

Mr. JOHN STEHN, University of Iowa, has been appointed instructor at the University of Oklahoma, and Mr. Stephan Brixey has been appointed assistant.

The following additional courses in Mathematics are announced for the summer of 1927:

**University of Colorado** First term, June 20 to July 23; Second term, July 25 to August 26. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, calculus, the following courses will be offered. First term: By Professor LIGHT, Teacher's Course in Mathematics; History of Mathematics; Functions of a complex variable. By Professor KEMPNER: Advanced Teacher's Course in Mathematics; Differential Equations; Theory of numbers. Second term: By Professor LIGHT: Statistics; Theory of Equations; Functions of a complex variable (continued). By Professor KEMPNER: Advanced Teacher's Course; Differential Equations (continued); Theory of numbers (continued).

The following 44 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1926; the university, month in which the degree was conferred, minor subjects (other than mathematics), and title of dissertation are given in each case if available.

EVELYN F. AYLESWORTH, California, May, *The dielectric constant of atomic hydrogen and the Stark effect.*

WEALTHY BABCOCK, Kansas, June, physics, *On the geometry associated with certain determinants with linear elements.*

H. W. BAILEY, Illinois, May, astronomy, *The summability of single and multiple Fourier series.*

R. W. BARNARD, Chicago, December, *The Fredholm theory of linear integral equations in general analysis for quaternionic-valued functions.*

MARTHA H. BARTON, Johns Hopkins, June, physics, *Some applications of the generalized Kronecker symbol.*

FLORENCE BLACK, Kansas, June, Physics, *A reduced system of differential equations for the invariants of ternary forms.*

H. L. BLACK, Illinois, May, physics, *A Cremona group isomorphic with the group of the twenty-seven lines on a cubic surface.*

E. T. BROWNE, Chicago, September, *Involutions that belong to a linear class.*

N. B. CONKWRIGHT, Illinois, May, astronomy, *The summability of Birkhoff series.*

A. E. COOPER, Chicago, June, *A topical history of the theory of quadratic residues.*

A. H. COPELAND, Harvard, *Studies on the gyroscope.*

C. M. CRAMLET, Washington, June, physics and astronomy, *Invariant tensors and their applications to the study of determinants and allied tensor functions.*

H. T. DAVIS, June, Wisconsin, physics, *An existence theorem for the characteristic numbers of a certain boundary value problem.*

M. S. DEMOS, Harvard, *The group characteristics of the general and special quaternary linear homogeneous congruence groups.*

R. D. DONER, Illinois, May, physics, *The determination of the Peirce and Scheffer's algebras of order eight.*

FAY FARNUM, Cornell, June, physics, *Triadic Cremona nets of plane curves.*

ORRIN FRINK, Columbia, December, *The operations of boolean algebras.*

R. J. GARVER, Chicago, September, I: *On Tschirnhaus transformations*; II *Division algebras of order sixteen.*

J. S. GEORGES, Chicago, September, *Associativity conditions for division algebras corresponding to any abelian group.*

R. F. GRAESSER, Illinois, May, statistics and physics, *A certain general type of Neumann expansions and expansions in confluent hyper-geometric functions.*

MARION C. GRAY, Bryn Mawr, June, physics, *Theory of singular ordinary differential equations of the second order.*

L. S. HILL, Yale, June, *Aggregate functions and an application in analysis situs.*

H. M. HOSFORD, Illinois, May, physics, *On the summability of Fourier-rational Bessel and Dini expansions.*

C. M. HUBER, Illinois, May, theoretical physics, *On complete systems of irrational invariants of associated point sets.*

F. E. JOHNSTON, Illinois, May, astronomy, *Transitive substitution groups containing a regular subgroup of lower degree.*

B. F. KIMBALL, Cornell, June, physics, *Geodesics on a toroid.*

F. W. KOKOMOOR, Michigan, June, physics, *The teaching of elementary geometry in the seventeenth century.*

B. O. KOOPMAN, Harvard, *On rejection to infinity and exterior motion in the restricted problem of three bodies.*

B. C. PATTERSON, Johns Hopkins, June, geophysics, *The algebraic and differential invariants of inversive geometry.*

H. R. PHALEN, Chicago, June, *Metric properties of the quadric of Moutard.*

I. R. POUNDER, Chicago, September, *A method of successive approximations for a partial equation of hyperbolic type.*

H. H. PRIDE, New York University, June, *On a certain quintic curve with a triple point.*

D. E. RICHMOND, Cornell, June, physics, *Geodesics on surfaces of genus zero with knobs.*

C. F. ROOS, Rice Institute, June, theoretical economics, *Generalized Lagrange problems in the calculus of variations.*

D. P. SCHWARTZ, Yale, June, *Studies in non-Euclidean geometry.*

M. M. SLOTNIK, Harvard, *Fundamental transformations of surfaces.*

H. L. SMITH, Chicago, September, *The Minkowski linear measure for a simple rectifiable curve.*

P. A. SMITH, Princeton, *Approximation of curves and surfaces by algebraic curves and surfaces.*

MARION E. STARK, Chicago, September, *A self-adjoint boundary value problem associated with a problem of the calculus of variations.*

GUY STEVENSON, Illinois, May, physics, *Expansions of the Neumann type in terms of products of Bessel functions.*

M. H. STONE, Harvard, *Ordinary linear homogeneous differential equations of order two and the related expansion problems.*

TAKASHA TERAMI, California, May, physics, *The solution of the differential equation of a vibrating membrane by successive approximations.*

E. T. VIRATA, Johns Hopkins, June, geophysics, *W-surfaces which have an isometric spherical representation of their lines of curvature.*

P. S. WAGNER, Johns Hopkins, June, geophysics, *An extension of Clifford's chain.*

Mr. A. C. BOSE, Deputy Magistrate, Bengal Provincial Civil Service, died December 11, 1926.

Father RIGGE of Creighton University, Omaha, Nebraska, died in March of this year.

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## DIRECTORY

**EDITORIAL CORRESPONDENCE** should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

**BOOKS FOR REVIEW** should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

**BUSINESS CORRESPONDENCE** should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

### MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.

Twelfth Annual Meeting, Nashville, Tenn., December, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 7.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 14.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November 26.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

## TWELFTH ANNUAL MEETING OF THE OHIO SECTION

The twelfth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 8, 1927, in connection with the meetings of the Ohio College Association. Chairman H. W. Kuhn presided over the afternoon session. Following the dinner in the evening there was a round table discussion under the direction of Professor I. A. Barnett, chairman of the program committee.

Fifty-five persons registered attendance, among whom were the following forty-three members of the Association: R. B. Allen, W. E. Anderson, Grace M. Bareis, I. A. Barnett, H. M. Beatty, H. Blumberg, R. L. Borger, L. Brand, J. B. Brandeberry, C. T. Bumer, R. S. Burington, V. B. Caris, E. H. Clarke, R. Crane, O. L. Dustheimer, C. A. Garabedian, H. Hancock, E. J. Hirschler, E. M. Justin, J. H. Kindle, H. W. Kuhn, C. C. MacDuffee, J. E. Merrill, C. C. Morris, R. L. Newlin, J. R. Overman, J. Pierce, S. E. Rasor, B. H. Redditt, Hortense Rickard, S. A. Rowland, E. C. Rupp, Mary E. Sinclair, S. A. Singer, A. Helen Tappan, J. C. Tinner, M. O. Tripp, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, P. D. Wilkins, J. B. Winslow, C. H. Yeaton.

Several items of business of local interest were transacted, and the following officers were elected for the coming year: Chairman, C. H. YEATON; Secretary-Treasurer, RUFUS CRANE; Member of the Executive Committee, C. T. BUMER; Member of the Program Committee, HENRY BLUMBERG. It is expected that the next meeting will be held at the Ohio State University on April 6, 1928.

The following five papers were presented:

1. "Galois fields and permutation groups" by the Chairman, Professor H. W. KUHN, Ohio State University.

2. "A new type of singular solution" by Professor H. W. SIBERT, University of Cincinatti, by invitation.

3. "The mystic number seven" by Professor HARRIS HANCOCK, University of Cincinnati.

4. "Mathematics as related to other collegiate work" by Professor O. L. DUSTHEIMER, Baldwin-Wallace College.

5. "The calculus of variations, from the standpoint of vector analysis" by Professor LOUIS BRAND, University of Cincinnati.

Abstracts of these papers follow, the numbers corresponding to those in the list of titles.

1. In this paper Professor Kuhn showed how two groups determined by the addition and the multiplication tables of a Galois field of order $2^3$ can be